MATH 185: COMPLEX ANALYSIS
FALL 2009/10
PROBLEM SET 1 SOLUTIONS

Throughout the problem set, $i = \sqrt{-1}$; and whenever we write $\alpha + \beta i$, it is implicit that $\alpha, \beta \in \mathbb{R}$.

1. Determine the values of the following (without the aid of any electronic devices).

(a) $(1+i)^{20} - (1-i)^{20}$.

SOLUTION. $(1+i)^{20} - (1-i)^{20} = [(1+i)^2]^{10} - [(1-i)^2]^{10} = (2i)^{10} - (-2i)^{10} = 0$.

(b) $\cos \frac{1}{4}\pi + i \cos \frac{3}{4}\pi + \cdots + i^n \cos(\frac{2n+1}{4})\pi + \cdots + i^{40} \cos \frac{81}{4}\pi$.

SOLUTION. Write $a_n = i^n \cos(\frac{2n+1}{4})\pi$. Note that

$$a_{n+2} = -i^n \cos\left[(\tfrac{2n+1}{4})\pi + \pi\right] = i^n \cos(\tfrac{2n+1}{4})\pi = a_n.$$

So $a_0 = a_2 = \cdots = a_{40}$, $a_1 = a_3 = \cdots = a_{39}$, and

$$a_0 + a_1 + \cdots + a_{40} = 21a_0 + 20a_1 = \tfrac{\sqrt{2}}{2}(21 - 20i).$$

(c) $1 + 2i + 3i^2 + \cdots + (m+1)i^m$ where m is divisible by 4.

SOLUTION. Let S be the sum. Then

$$\begin{aligned} S &= 1 + 2i + 3i^2 + \cdots + (m+1)i^m, \\ iS &= i + 2i^2 + \cdots + mi^m + (m+1)i^{m+1}. \end{aligned}$$

Subtracting the second equation from the first yields

$$\begin{aligned} (1-i)S &= 1 + i + i^2 + \cdots + i^m - (m+1)i^{m+1} \\ &= \frac{1 - i^{m+1}}{1-i} - (m+1)i^{m+1} \\ &= 1 - (m+1)i \end{aligned}$$

since $i^m = 1$ if m is divisible by 4. Hence

$$S = \frac{1 - (m+1)i}{1-i} \times \frac{1+i}{1+i} = \frac{1}{2}(m + 2 - mi).$$

2. Use the exponential form of $\cos\theta$ and $\sin\theta$ to show the following.

(a) Show that

$$1 + n\cos\theta + \cdots + \frac{n!}{r!(n-r)!}\cos r\theta + \cdots + \cos n\theta = (2\cos\tfrac{1}{2}\theta)^n \cos\tfrac{1}{2}n\theta.$$

Prove that, as $n \to \infty$, the series converges to 0 if $\frac{2}{3}\pi < \theta < \frac{4}{3}\pi$.

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Solution. Using the form $\cos r\theta = \frac{1}{2}(e^{ir\theta} + e^{-ir\theta})$, series becomes

$$\begin{aligned}
\sum_{r=1}^{n} \binom{n}{r} \cos r\theta &= \frac{1}{2}\sum_{r=1}^{n} \binom{n}{r} e^{ir\theta} + \frac{1}{2}\sum_{r=1}^{n} \binom{n}{r} e^{-ir\theta} \\
&= \frac{1}{2}(1 + e^{i\theta})^n + \frac{1}{2}(1 + e^{-i\theta})^n \\
&= \frac{1}{2}e^{i\frac{1}{2}n\theta}(e^{-i\frac{1}{2}\theta} + e^{i\frac{1}{2}\theta})^n + \frac{1}{2}e^{-i\frac{1}{2}n\theta}(e^{i\frac{1}{2}\theta} + e^{-i\frac{1}{2}\theta})^n \\
&= \frac{1}{2}(e^{i\frac{1}{2}n\theta} + e^{-i\frac{1}{2}n\theta})(e^{-i\frac{1}{2}\theta} + e^{i\frac{1}{2}\theta})^n \\
&= \cos \tfrac{1}{2}n\theta(2\cos\tfrac{1}{2}\theta)^n.
\end{aligned}$$

Note that $(2\cos\frac{1}{2}\theta)^n \cos\frac{1}{2}n\theta \to 0$ if $(2\cos\frac{1}{2}\theta)^n \to 0$, i.e. if $|\cos\frac{1}{2}\theta| < \frac{1}{2}$. So when $\frac{2}{3}\pi < \theta < \frac{4}{3}\pi$, then $\frac{1}{3}\pi < \frac{1}{2}\theta < \frac{2}{3}\pi$ and $-\frac{1}{2} < \cos\frac{1}{2}\theta < \frac{1}{2}$ as required.

(b) If $\sin\theta = \alpha\sin(\theta + \beta)$, where α and β are real constants, prove that

$$e^{2i\theta} = \frac{1 - \alpha e^{-i\beta}}{1 - \alpha e^{i\beta}}.$$

Hence prove that

$$\theta = \sum_{n=1}^{\infty} \frac{\alpha^n}{n} \sin n\beta.$$

State the range of values of α for which the series is valid.

Solution. Written in exponential form, $\sin\theta = \alpha\sin(\theta + \beta)$ becomes

$$\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{\alpha}{2i}(e^{i(\theta+\beta)} - e^{-i(\theta+\beta)}).$$

Multiplying both sides by $2ie^{i\theta}$ gives

$$e^{2i\theta} - 1 = \alpha(e^{2i\theta + i\beta} - e^{-i\beta})$$

and thus

$$e^{2i\theta} = \frac{1 - \alpha e^{-i\beta}}{1 - \alpha e^{i\beta}}.$$

Taking logarithms, we get

$$\begin{aligned}
2i\theta &= \log(1 - \alpha e^{-i\beta}) - \log(1 - \alpha e^{i\beta}) \\
&= -\left[\alpha e^{-i\beta} + \frac{\alpha^2}{2}e^{-2i\beta} + \frac{\alpha^3}{3}e^{-3i\beta} + \cdots\right] + \left[\alpha e^{i\beta} + \frac{\alpha^2}{2}e^{2i\beta} + \frac{\alpha^3}{3}e^{3i\beta} + \cdots\right]. \\
\theta &= \frac{1}{2i}\left[\alpha(e^{i\beta} - e^{-i\alpha}) + \frac{\alpha^2}{2}(e^{2i\beta} - e^{-2i\beta}) + \cdots\right] \\
&= \sum_{n=1}^{\infty} \frac{\alpha^n}{n} \sin n\beta.
\end{aligned}$$

The logarithmic expansions are valid provided $|\alpha e^{\pm i\beta}| < 1$, ie. $|\alpha| < 1$. In which case, the series is absolutely convergent and thus justifying the rearrangement of terms.

3. Express the roots of the equation $z^7 - 1 = 0$ in the form $\cos\theta + i\sin\theta$. Hence show that the roots of the equation

$$u^3 + u^2 - 2u - 1 = 0$$

are

$$2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2\cos\frac{6\pi}{7},$$

and find the roots of

$$8w^3 + 4w^2 - 4w - 1 = 0.$$

Solution. The roots of $z^7 - 1 = 0$ are

$$e^{\frac{2n\pi i}{7}} = \cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}, \quad n = 0, \pm1, \pm2, \pm3.$$

Factoring out $z - 1$ (corresponding to $n = 0$) gives

$$z^7 - 1 = (z-1)(z^6 + z^5 + \cdots + z + 1).$$

The degree 6 polynomial must then be the product of the remaining factors:

$$\begin{aligned} z^6 + z^5 + \cdots + z + 1 &= \prod_{n=-3}^{3} (z - e^{\frac{2n\pi i}{7}}) \\ &= (z^2 - \alpha z + 1)(z^2 - \beta z + 1)(z^2 - \gamma z + 1) \end{aligned}$$

where the three quadratic factors are the products of the three conjugate pairs of linear factors and

$$\alpha = 2\cos\frac{2\pi}{7}, \quad \beta = 2\cos\frac{4\pi}{7}, \quad \gamma = 2\cos\frac{6\pi}{7}.$$

Expanding the product of the three quadratic factors and comparing coefficients with the degree 6 polynomial gives

$$\alpha + \beta + \gamma = -1, \quad \alpha\beta + \beta\gamma + \gamma\alpha = -2, \quad \alpha\beta\gamma = -1.$$

We notice that the roots of the equation $u^3 + u^2 - 2u - 1 = 0$ satisfy precisely these relations — which in turn allows us to deduce that they are α, β, γ. For the last part, we substitute $u = 2w$ to get the equation in w and to see that its roots are

$$\cos\frac{2\pi}{7}, \quad \cos\frac{4\pi}{7}, \quad \cos\frac{6\pi}{7}.$$

4. (a) Find all possible values of i^i. Express your solutions in the form $\alpha + \beta i$.
Solution. $i = e^{\pi i/2 + 2n\pi i}$, $n \in \mathbb{Z}$. $i^i = e^{-\pi/2 - 2n\pi}$, $n \in \mathbb{Z}$. [We will see a more rigorous treatment later in the course.]

(b) Find all values of $\theta \in [0, 2\pi)$ for which the following limit exists

$$\lim_{r\to\infty} e^{re^{i\theta}}.$$

Solution. Note that $e^{re^{i\theta}} = e^{r\cos\theta}e^{ir\sin\theta}$. If the limit exists, then so must

$$\lim_{r\to\infty} |e^{re^{i\theta}}| = \lim_{r\to\infty} e^{r\cos\theta}.$$

Hence we must have $\cos\theta \le 0$. But if $\cos\theta = 0$, then $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ and it is easy to see that $e^{re^{i\theta}} = e^{\pm ir}$ does not have a limit as $r \to \infty$ (e.g. take a sequence $r_n = n\pi$). So we must have $\cos\theta < 0$, i.e. $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

5. Let $a_0, \ldots, a_4 \in \mathbb{R}$. Suppose the polynomial equation

$$a_4 z^4 + ia_3 z^3 + a_2 z^2 + ia_1 z + a_0 = 0$$

has a root given by $z = \alpha + \beta i$. Find another root of the equation. Your answer should only depend on α, β.
Solution. Taking complex conjugate of the equation, we get

$$a_4 \bar{z}^4 - ia_3 \bar{z}^3 + a_2 \bar{z}^2 - ia_1 \bar{z} + a_0 = 0.$$

Now observe that this may be rewritten as

$$a_4(-\bar{z})^4 + ia_3(-\bar{z})^3 + a_2(-\bar{z})^2 + ia_1(-\bar{z}) + a_0 = 0.$$

In other words, if $z = \alpha + \beta i$ is a root, then so is $-\bar{z} = -\alpha + \beta i$.

6. Let $z_n, w_n \in \mathbb{C}$ for every $n \in \mathbb{N}$. Show that

(a) If $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ are both convergent, then so is

$$\sum_{n=1}^{\infty} \lambda z_n + \mu w_n$$

for any $\lambda, \mu \in \mathbb{C}$.

(b) If $\sum_{n=1}^{\infty} z_n$ is convergent, then

$$\lim_{n \to \infty} z_n = 0.$$

(c) If $\sum_{n=1}^{\infty} |z_n|$ is convergent, then so is $\sum_{n=1}^{\infty} z_n$.

SOLUTION. Routine exercise.