MATH 185: COMPLEX ANALYSIS
FALL 2009/10
PROBLEM SET 1

Throughout the problem set, $i = \sqrt{-1}$; and whenever we write $\alpha + \beta i$, it is implicit that $\alpha, \beta \in \mathbb{R}$.

1. Determine the values of the following (without the aid of any electronic devices).
(a) $(1+i)^{20} - (1-i)^{20}$.
(b) $\cos \frac{1}{4}\pi + i \cos \frac{3}{4}\pi + \cdots + i^n \cos(\frac{2n+1}{4})\pi + \cdots + i^{40} \cos \frac{81}{4}\pi$.
(c) $1 + 2i + 3i^2 + \cdots + (m+1)i^m$ where m is divisible by 4.

2. Use the exponential form of $\cos\theta$ and $\sin\theta$ to show the following.
(a) Show that

$$1 + n\cos\theta + \cdots + \frac{n!}{r!(n-r)!}\cos r\theta + \cdots + \cos n\theta = (2\cos\tfrac{1}{2}\theta)^n \cos \tfrac{1}{2}n\theta.$$

Prove that, as $n \to \infty$, the series converges to 0 if $\frac{2}{3}\pi < \theta < \frac{4}{3}\pi$.
(b) If $\sin\theta = \alpha\sin(\theta + \beta)$, where α and β are real constants, prove that

$$e^{2i\theta} = \frac{1 - \alpha e^{-i\beta}}{1 - \alpha e^{i\beta}}.$$

Hence prove that

$$\theta = \sum_{n=1}^{\infty} \frac{\alpha^n}{n} \sin n\beta.$$

State the range of values of α for which the series is valid.

3. Express the roots of the equation $z^7 - 1 = 0$ in the form $\cos\theta + i\sin\theta$. Hence show that the roots of the equation

$$u^3 + u^2 - 2u - 1 = 0$$

are

$$2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2\cos\frac{6\pi}{7},$$

and find the roots of

$$8w^3 + 4w^2 - 4w - 1 = 0.$$

4. (a) Find all possible values of i^i. Express your solutions in the form $\alpha + \beta i$.
(b) Find all values of $\theta \in [0, 2\pi)$ for which the following limit exists

$$\lim_{r\to\infty} e^{re^{i\theta}}.$$

5. Let $a_0, \dots, a_4 \in \mathbb{R}$. Suppose the polynomial equation

$$a_4 z^4 + i a_3 z^3 + a_2 z^2 + i a_1 z + a_0 = 0$$

has a root given by $z = \alpha + \beta i$. Find another root of the equation. Your answer should only depend on α, β.

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6. Let $z_n, w_n \in \mathbb{C}$ for every $n \in \mathbb{N}$. Show that

(a) If $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ are both convergent, then so is

$$\sum_{n=1}^{\infty} \lambda z_n + \mu w_n$$

for any $\lambda, \mu \in \mathbb{C}$.

(b) If $\sum_{n=1}^{\infty} z_n$ is convergent, then

$$\lim_{n \to \infty} z_n = 0.$$

(c) If $\sum_{n=1}^{\infty} |z_n|$ is convergent, then so is $\sum_{n=1}^{\infty} z_n$.