

Solutions to Math 1a section worksheet, 12/9/04

1. The region looks like a smiley face. To generate the solid, we only need the right half of the region, which is bounded by the two curves and the  $y$ -axis. The half region contains the points  $(0, 0)$ ,  $(0, 4)$ ,  $(2, 8)$  on the boundary.

(a) The horizontal slices are circles when  $0 \leq y \leq 4$  but are washers when  $4 < y \leq 8$ . To determine various radii, we need to express everything in terms of  $y$ . The outer curve is  $y = 2x^2$ , which becomes  $x = \sqrt{y/2}$  (not  $x = -\sqrt{y/2}$  because we took the right half of curve). The inner curve is  $y = x^2 + 4$ , which becomes  $x = \sqrt{y-4}$ . Thus

$$\begin{aligned} \text{volume} &= \int_0^4 \pi(\sqrt{y/2})^2 dy + \int_4^8 \pi(\sqrt{y/2})^2 - \pi(\sqrt{y-4})^2 dy \\ &= \int_0^4 \pi \frac{y}{2} dy + \int_4^8 \pi \frac{y}{2} dy - \int_4^8 \pi(y-4) dy \\ &= \int_0^8 \pi \frac{y}{2} dy - \int_4^8 \pi(y-4) dy \\ &= \pi \left[ \frac{1}{4} y^2 \right]_0^8 - \pi \left[ \frac{1}{2} y^2 - 4y \right]_4^8 \\ &= \pi(16 - 0) - \pi(0 - (-8)) \\ &= 8\pi. \end{aligned}$$

(b) The cylindrical shell at  $x$  has height  $x^2 + 4 - 2x^2 = -x^2 + 4$  and radius  $x$  (due to being at a distance  $x - 0$  from the  $y$  axis). Thus its volume is  $2\pi x(-x^2 + 4)$ , and

$$\begin{aligned} \text{volume} &= \int_0^2 2\pi(-x^3 + 4x) dx \\ &= 2\pi \left[ -\frac{1}{4} x^4 + 2x^2 \right]_0^2 \\ &= 2\pi(-4 + 8) \\ &= 8\pi. \end{aligned}$$

2. It turns out that we did a similar example in lecture on Friday, 12/10/04. The remaining piece of the earth can be considered to be the solid generated by rotating the region bounded by  $x = \sqrt{r^2 - y^2}$  (the right half of the circle),  $y = r/2$ , and the  $y$ -axis around the  $y$ -axis. If we take horizontal slices, at  $y$  the slice is a disk with radius  $\sqrt{r^2 - y^2}$  and hence area  $\pi(r^2 - y^2)$ . The minimum and maximum  $y$  values are  $-r$  and  $r/2$ . Hence

$$\begin{aligned} \text{volume} &= \int_{-r}^{\frac{r}{2}} \pi(r^2 - y^2) dy \\ &= \pi \left[ r^2 y - \frac{1}{3} y^3 \right]_{-r}^{\frac{r}{2}} \\ &= \pi \left( \frac{r^3}{2} - \frac{r^3}{24} + r^3 - \frac{r^3}{3} \right) \\ &= \frac{9}{8} \pi r^3. \end{aligned}$$

3. The cylindrical shell at  $x$  has height  $\sqrt{x} - x$  and radius  $x$  (due to being at a distance  $x - 0$  from the  $y$  axis). Thus its volume is  $2\pi x(\sqrt{x} - x)$ . The region being rotated has minimum  $x$  value 0 and maximum  $x$  value 1. Hence

$$\begin{aligned} \text{volume} &= \int_0^1 2\pi(x^{3/2} - x^2)dx \\ &= 2\pi \left[ \frac{2}{5}x^{5/2} - \frac{1}{3}x^3 \right]_0^1 \\ &= 2\pi \left( \frac{2}{5} - \frac{1}{3} \right) \\ &= \frac{2}{15}\pi. \end{aligned}$$

4. We take vertical slices perpendicular to the  $x$ -axis. When  $0 \leq x < 2$ , the slice is a washer with inner radius  $4 - x^2$  and outer radius  $4 - 0 = 4$ . When  $2 \leq x \leq 3$ , the part of the region above  $y = 4$  produces a disk of radius  $x^2 - 4$ , while the part below  $y = 4$  produces a disk of radius 4. The actual cross-section superimposes these two disks, giving us the larger one. When  $2 \leq x \leq \sqrt{8} = 2\sqrt{2}$ ,  $x^2 - 4 \leq 4$  and we get a disk of radius 4. When  $2\sqrt{2} \leq x \leq 3$ ,  $x^2 - 4 \geq 4$  and we get a disk of radius  $x^2 - 4$ . Therefore,

$$\text{volume} = \int_0^2 \pi(4^2 - (4 - x^2)^2)dx + \int_2^{2\sqrt{2}} \pi(4^2)dx + \int_{2\sqrt{2}}^3 \pi(x^2 - 4)^2dx = \frac{113 + 256\sqrt{2}}{15}\pi.$$

The point of the problem is setting up the integrals, not the final, messy computation.

5. Recall that the volume  $V$  of the solid donut is  $2\pi^2 R^2 r^2$ .
- (a) When we apply the glaze,  $R$  remains the same but  $r$  changes, so let us think of the volume  $V$  as a function of  $r$ , keeping  $R$  constant. Then the volume of glaze is the change in  $V$ , which is  $V(r + dr) - V(r)$ . Since  $dr$  is small, we can approximate this difference as  $V'(r)dr$  (where we differentiated  $V$  with respect to  $r$ ). Since  $V = 2\pi^2 R^2 r^2$ , volume of glaze  $\approx V'(r)dr = 4\pi^2 R r dr$ .
- (b) Let  $f(r)$  be the profit function. Then  $f(r) = P(R + r) - 4\pi^2 R G r - 2\pi^2 R D r^2$ . The domain is  $[0, R]$ . Now  $f'(r) = P - 4\pi^2 R G - 4\pi^2 R D r$ , so  $f'(r) = 0$  when  $r = \frac{P - 4\pi^2 R G}{4\pi^2 R D}$ . For convenience, denote  $\frac{P - 4\pi^2 R G}{4\pi^2 R D}$  as  $C$ . Note that  $C$  is positive because we assumed  $P - 4\pi^2 R G > 0$ , and the denominator is positive too. This critical point  $C$  may or may not appear in the domain.
- If  $C \leq R$ , then  $C$  is in the domain. Now  $f'(0) = P - 4\pi^2 R G > 0$ , so  $f' > 0$  on  $(0, C)$ . Since  $f'(r)$  is a linear function of  $r$ , it must change signs at the root  $C$ , so  $f' < 0$  on  $(C, R)$ . This means that the graph of  $f$  is an upside-down  $V$  shape, so  $f$  is maximized at  $r = C = \frac{P - 4\pi^2 R G}{4\pi^2 R D}$ .
- If  $C > R$ , then the domain is entirely on one side of  $C$ , so  $f'$  must have the same sign in the entire domain. Again  $f'(0) = P - 4\pi^2 R G > 0$ , so the sign is positive and  $f' > 0$  throughout  $[0, R]$ . In other words,  $f$  is increasing on the domain, so it is maximized at  $r = R$ .
- In summary, if  $\frac{P - 4\pi^2 R G}{4\pi^2 R D} \leq R$ , then the profit is maximized at  $r = \frac{P - 4\pi^2 R G}{4\pi^2 R D}$ . Otherwise, the profit is maximized at  $r = R$ .