C.21 Semilocal Rings

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From Commutative to Noncommutative

In commutative algebra, a ring R is said to be *local* if it has a unique maximal ideal, and *semilocal* if it has finitely many maximal ideals. If $\mathfrak{p}$ is a prime ideal in a commutative ring, we can "invert" the elements in $S = R \setminus \mathfrak{p}$ to form a local ring R_S, with the unique maximal ideal $\mathfrak{p}R_S$. More generally, if $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are prime ideals in R, then we can "invert" the elements in $S = R \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n)$ to form a semilocal ring R_S, with maximal ideals among $\mathfrak{p}_1 R_S, \ldots, \mathfrak{p}_n R_S$. Thus, for instance, all rational numbers r/s with s prime to 30 form a semilocal ring, with maximal ideals generated respectively by 2, 3, and 5. This is a PID; as a matter of fact, any semilocal Dedekind domain is a PID. As a second instance of "semilocalization", note that if R is a commutative noetherian ring, then the set of zero-divisors is a union of finitely many prime ideals (namely, the "associated primes" of (0)). Thus, we obtain:

C.21.1. Theorem. *If R is a commutative noetherian ring, then its classical ring of quotients (obtained from R by inverting all of its non zero-divisors) is a semilocal ring.*

How about noncommutative rings? We have seen in the last section of C.20 that local rings make sense in general: they are rings with a unique maximal left (or right) ideal. Some facts on commutative local rings generalize nicely to general local rings. The case of semilocal rings is, however, more tricky. The correct definition was first given by H. Bass [1]: *a ring R is semilocal if $R/\mathrm{rad}(R)$ is a left (or right) artinian ring* (where $\mathrm{rad}(R)$ denotes the Jacobson radical of R: see C.4). This condition turns out to be *weaker* (in the noncommutative case) than R having only finitely many maximal left (right) ideals. The case of *local rings* corresponds to the case where $R/\mathrm{rad}(R)$ is a division ring.

Bass's definition was perhaps partly motivated by the following more classical definition: R is a *semiprimary ring* if $R/\mathrm{rad}(R)$ is artinian and $\mathrm{rad}(R)$ is nilpotent. Take away the latter condition and you get the definition of a semilocal ring. In particular, left (right) artinian rings are semilocal, which generalizes nicely the fact that commutative artinian rings have a finite number of maximal ideals. Other useful classes of semilocal rings include, for instance, finite direct products of local rings, matrix rings over local rings, and module-finite algebras over commutative semilocal rings.

Key Results on Semilocal Rings

In studying the stable structure of general linear groups in algebraic K-theory, Bass proved the following basic result (ca. 1964) on the unit structure of semilocal rings.

C.21.2. Theorem. *If R is a semilocal ring, then R has stable range 1, in the sense that, whenever $Ra + Rb = R$, there exists $r \in R$ such that $a + rb \in U(R)$.*

This theorem acquired added significance after E. G. Evans proved (ca. 1973) the following remarkable result on the cancellation of modules from direct sums.

C.21.3. Theorem. *Let A be a ring, and M, P, Q be right A-modules. If $E = \mathrm{End}_A(M)$ has stable range 1 (e.g. E is semilocal), then $M \oplus P \cong M \oplus Q \implies P \cong Q$.*

In 1992, R. Camps and W. Dicks [2] obtained a number of surprising new characterizations of semilocal rings. We shall state here only two of their many characterizations.

C.21.4. Theorem. *The following conditions are each equivalent to R being semilocal:*
(1) There exists an (artinian) semisimple ring S and a ring homomorphism $f : R \to S$ such that $f^{-1}(U(S)) \subseteq U(R)$;
(2) There exists a non-negative integer n and a function $d : R \to \{0, 1, \ldots, n\}$ such that, for all a, $b \in R$, $d(1 - ab) + d(a) = d(a - aba)$, and $d(a) = 0$ only if $a \in U(R)$.

Three remarkable consequences of the Camps-Dicks Theorem in [2] are recorded below. The first of these confirmed a conjecture of P. Menal, and was largely motivated by the classical fact that the endomorphism ring of a module of finite length is semiprimary.

C.21.5. Corollary. *If M is an artinian right module over a ring A, then $\mathrm{End}_A(M)$ is semilocal. In particular, M cancels from direct sums in the category of right A-modules.*

C.21.6. Corollary. *If R is a subring of a left artinian ring R' such that R'/R is artinian as a left R-module, then R is a semilocal ring.*

The third consequence of C.21.4 below is a generalization of C.21.1 to the noncommutative case. It was first proved by J. T. Stafford for left and right noetherian rings, and proved in the following form by F. Cedó as a consequence of the main results in [2].

C.21.7. Corollary. *Let R be a "classical quotient ring" in the sense that every non zero-divisor in R is a unit. If R satisfies ACC on left as well as on right annihilators, then R is a semilocal ring.*

For modules with semilocal endomorphism rings, Facchini, Herbera, Levy, and Vámos [3] proved the following "n-Cancellation" result for any positive integer n.

C.21.8. Theorem. *Let M, N be right A-modules, with $\mathrm{End}_A(M)$ semilocal. If $M \oplus \cdots \oplus M \cong N \oplus \cdots \oplus N$ (n copies on each side), then $M \cong N$.*

The results C.21.5 and C.21.8 generated a whole lot of new interest on the class $\mathcal{S}(A)$ of right A-modules whose endomorphism rings are semilocal. A general result in this direction is that of Herbera-Shamsuddin [5], which states that *A-modules of finite uniform and co-uniform dimensions belong to $\mathcal{S}(A)$.* (A module is said to have finite uniform dimension if it contains no infinite independent set of nonzero submodules, and finite co-uniform dimension if it contains no infinite co-independent set of proper submodules.) This implies, for instance, that any linearly compact A-module is in $\mathcal{S}(A)$, as is any finite direct sum of uniserial A-modules. For the terminology used here, as well as an excellent presentation of the results surveyed in this section, see Facchini's recent monograph [4].

Special Classes of Semilocal Rings

For more structural information and module-theoretic results to be available for semilocal rings, we often specialize to suitable subclasses of such rings. Foremost among these are the perfect and semiperfect rings introduced by Bass (ca. 1960) as "homological generalizations" of the semiprimary rings.

Recall that a *projective cover* for a module M is a projective module P with an epimorphism $f : P \to M$ such that $\ker(f)$ is "small" (that is, for any submodule $S \subseteq P$, $S + \ker(f) = P$ implies $S = P$). Such a projective cover may not exist for a given M, but if it exists, then it is unique (over M). (The definition of a projective cover was largely inspired by that of an injective hull.) According to Bass, a ring R is *left perfect* if every left R-module has a projective cover, and *left semiperfect* if every finitely generated (or just cyclic) left R-module has a projective cover.

It turns out that "semiperfect" is a left-right symmetric notion, so we may just speak of semiperfect rings. Bass proved that these are precisely those semilocal rings R for which idempotents in $R/\mathrm{rad}(R)$ can be lifted to idempotents in R. [In the commutative category, semiperfect rings are precisely finite direct products of (commutative) local rings.] For a right module M over any ring A, there is a simple criterion for $\mathrm{End}_A(M)$ to be semiperfect: this is the case iff M has a decomposition $M_1 \oplus \cdots \oplus M_n$ where each M_i is *strongly indecomposable*, that is, $\mathrm{End}_A(M_i)$ is local. In particular, such a direct sum always belongs to $\mathcal{S}(A)$.

Contrary to the case of semiperfect rings, left perfect rings need not be right perfect. Bass proved that left perfect rings are precisely those semilocal rings R for which $\mathrm{rad}(R)$ is "left T-nilpotent", that is, for any sequence $a_1, a_2, \ldots \in \mathrm{rad}(R)$, $a_1 a_2 \cdots a_n = 0$ for some n; or equivalently, those semilocal rings R for which every nonzero left R-module has a maximal submodule. Amazingly, left perfect rings can also be characterized by right-sided conditions! Indeed, one of Bass's most beautiful results is that a ring is *left* perfect iff it satisfies DCC on principal *right* ideals. Detailed proofs of all of the above results can be found in my introductory text [6] (Chapter 8).

Since their inception in the early 1960s, perfect and semiperfect rings have permeated noncommutative ring theory, and have been used extensively in the research on Krull-Schmidt theory, Morita duality theory, representation theory, model-theoretic algebra, as well as on self-injective rings, exchange rings, cogenerator rings, CS rings, *inter alia.* For a more detailed survey on these applications of perfect and semiperfect rings, see [7].

References

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