Projective Modules over Some Prüfer Rings

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§1. Introduction

This note is a contribution to the theory of Prüfer rings with 0-divisors. For certain classes of Prüfer rings R, we classify the f.g. (finitely generated) projective R-modules, compute the reduced Grothendieck group $\tilde{K}_0(R)$, and show that non countably generated projective R-modules are free.

All rings R in this note will be assumed to be commutative with an identity element. The set of 0-divisors in R is denoted by $\mathcal{Z}(R)$; for convenience, we take 0 to be an element of $\mathcal{Z}(R)$. Elements in $R \setminus \mathcal{Z}(R)$ (the non 0-divisors) are said to be *regular*; an ideal in R is said to be regular if it contains a regular element of R. By definition, a Prüfer ring is a (commutative) ring in which every f.g. regular ideal is invertible. Many characterizations of Prüfer rings are known; see, e.g. [Gr] and [LM].

The theory of Prüfer *domains* is by now very well developed. For an excellent presentation of this theory, see the recent monograph by Fontana, Huckaba, and Papick [FHP]. Since Prüfer domains are precisely the semihereditary domains[1], a classical theorem of Cartan and Eilenberg [CE: Ch. I, Prop. 6.1] implies that, over such rings, every f.g. projective module is isomorphic to a direct sum of (f.g. projective) ideals. Such a result is, however, not available over an arbitrary Prüfer ring R, since R may no longer be semihereditary. In Theorem 3 of this note, we'll show that, by imposing a "smallness" condition on the set of 0-divisors $\mathcal{Z}(R)$, we can restore a decomposition theorem on the f.g. projective modules over a Prüfer ring R. By definition, a 0-divisor $a \in \mathcal{Z}(R)$ is *small* if Ra is a small ideal, that is, for any ideal $A \subseteq R$, $A + Ra = R$ implies that $A = R$. We say that R *has small 0-divisors* if all $a \in \mathcal{Z}(R)$ are small. (For instance, integral domains and local rings have small 0-divisors.)

In the classical case of Dedekind domains, it is well known that the decomposition of f.g. projective modules into ideals is not unique, but the degree of nonuniqueness is completely controlled by the Steinitz Isomorphism Theorem; see, e.g. [Ka_2]. In [HL], Heitmann and Levy have obtained a generalization of the Steinitz Isomorphism Theorem to Prüfer domains having the so-called $1\frac{1}{2}$ generator property. As a consequence, the usual classification results for f.g. projective modules over Dedekind domains extend smoothly to this class of Prüfer domains. In order to further extend these results to rings with 0-divisors, we make the following modification of the definition given by Heitmann and

[1] A ring is called *hereditary* if all ideals are projective, and *semihereditary* if all f.g. ideals are projective.

Levy: a ring R is said to have the $1\frac{1}{2}$ *generator property* if, for any invertible ideal $I \subseteq R$ and any regular element $a \in I \setminus (\operatorname{rad} R)I$, there exists an element b such that $I = Ra + Rb$. (Here and in the following, $\operatorname{rad}(R)$ denotes the Jacobson radical of the ring R.) In §2, generalizing the work in [HL], we prove the Steinitz Isomorphism Theorem for this class of rings, again by using a smallness assumption on $\mathcal{Z}(R)$.

In the context of Prüfer rings (and in the language of algebraic K-theory), the first principal result in this paper can be stated as follows.

Theorem A. *Let R be a Prüfer ring with the $1\frac{1}{2}$ generator property having small 0-divisors. Then $\tilde{K}_0(R) \cong \operatorname{Pic}(R)$. (Here, $\tilde{K}_0(R) := K_0(R)/\mathbb{Z}\cdot[R]$, and $\operatorname{Pic}(R)$ denotes the Picard group of R.)*

The second principal result concerns projective modules that are not necessarily f.g. over Prüfer rings. Using some of the methods of Kaplansky [Ka_2, Ka_3] and Bass [Ba], we obtain the following.

Theorem B. *Let R be a Prüfer ring with small 0-divisors. Then a projective R-module is indecomposable iff it is isomorphic to an invertible ideal of R. If, moreover, R has the $1\frac{1}{2}$ generator property, then any infinite direct sum of nonzero countably generated projective R-modules is free, and any non countably generated projective R-module is free.*

In the course of proving these results, we have also generalized several known theorems in the literature from domains to arbitrary (commutative) rings; see §4.

§2. A General Steinitz Isomorphism Theorem

The classical Steinitz Isomorphism Theorem states that, for any nonzero ideals I, J in a Dedekind domain R, there is an R-isomorphism $I \oplus J \cong R \oplus IJ$. In view of the great importance of this theorem on the structure of f.g. modules over Dedekind rings, it is natural to try to extend the result to, say, invertible ideals over other classes of rings. In this section, we generalize the work of Heitmann and Levy [HL], and carry out the extension of the Steinitz Isomorphism Theorem to rings with the $1\frac{1}{2}$ generator property having small 0-divisors.

We begin with the following lemma that offers several characterizations for rings having small 0-divisors. For such rings, this result shows in particular that the regularity assumption on the element a in the definition of the $1\frac{1}{2}$ generator property can actually be dropped.

Lemma 1. *For any ring R, the following are equivalent:*

(1) *R has small 0-divisors.*
(2) $\mathcal{Z}(R) \subseteq \operatorname{rad}(R)$.
(3) $1 + \mathcal{Z}(R) \subseteq U(R)$ *(the group of units of R).*
(4) *For any invertible ideal $I \subseteq R$, any element $a \in I \setminus (\operatorname{rad} R)I$ is regular.*

Proof. (1) $\iff$ (2) is clear from the fact that $\operatorname{rad}(R)$ is the largest small ideal in R. Next, (2) $\iff$ (3) follows from the fact that $\mathcal{Z}(R)$ is closed under multiplication by elements of R, and the fact that an element $a \in R$ lies in $\operatorname{rad}(R)$ iff $1 + Ra \subseteq U(R)$.

(4) $\Longrightarrow$ (2). If $\mathcal{Z}(R) \nsubseteq \operatorname{rad}(R)$, fix an element $a \in \mathcal{Z}(R) \setminus \operatorname{rad}(R)$. Then for the invertible ideal $I = R$, the element $a \in I \setminus (\operatorname{rad} R)I$ fails to be regular.

(2) $\Longrightarrow$ (4). Assume that $\mathcal{Z}(R) \subseteq \operatorname{rad}(R)$, and consider an element a as in (4). Let $Q(R)$ be the total ring of quotients of R, and let

$$I^{-1} = \{q \in Q(R) : \ qI \subseteq R\}.$$

Since $I\,I^{-1} = R$, $a \in I \setminus (\operatorname{rad} R)I$ implies that $aI^{-1} \nsubseteq \operatorname{rad}(R)$, and of course $aI^{-1} \subseteq R$. If a is not regular, then $ra = 0$ for some nonzero $r \in R$. But then $r(aI^{-1}) = 0$ and so $aI^{-1} \subseteq \mathcal{Z}(R) \subseteq \operatorname{rad}(R)$, which is not the case. Thus, a must be regular. $\square$

The characterizations in Lemma 1 enable us to give some quick examples of rings with small 0-divisors besides obvious ones such as integral domains and local rings. For instance, any ring R whose 0-divisors are nilpotent is a ring with small 0-divisors. This is the class of rings expressible as A/Q where Q is a primary ideal in the ring A.

Rings having small 0-divisors, as characterized in (2) above, were first used systematically by Kaplansky in [Ka$_1$]; see his theorems (3.2), (5.1), (12.1), etc., which all depended on the property (2). It is perhaps tempting to think that there is a some kind of "duality" between the property (1) "0-divisors in R are small" and the property "non 0-divisors in R are large" (where "large" is in the sense of "essential"). However, the latter is true in *any* commutative ring, while the former is only a condition.

Lemma 2. *Let R be a ring with small 0-divisors, and I be an invertible ideal in R. Then:*

(1) *$I = Ra_1 + \cdots + Ra_n$, where each $a_i \in I \setminus (\operatorname{rad} R)I$ (and hence regular by Lemma 1).*

(2) *If, moreover, R has the $1\frac{1}{2}$ generator property, then $I = Ra + Rc$, where $a, c \in I \setminus (\operatorname{rad} R)I$ (and hence regular by Lemma 1), and a can be arbitrarily prescribed.*

Proof. (1) It is well known that I is finitely generated, say $I = Ra_1 + \cdots + Ra_m$. After relabelling the indices, we may assume that $a_1, \ldots, a_n \notin (\operatorname{rad} R)I$, while $a_{n+1}, \ldots, a_m \in (\operatorname{rad} R)I$. Then we have

$$I = Ra_1 + \cdots + Ra_n + (\operatorname{rad} R)I,$$

and so Nakayama's Lemma implies that $I = Ra_1 + \cdots + Ra_n$, as desired.[2]

(2) Start with any element $a \in I \setminus (\operatorname{rad} R)I$. Since a is automatically regular by the condition (4) in Lemma 1, the definition of the $1\frac{1}{2}$ generator property guarantees that

[2] In connection to the conclusion of this part, it is relevant to point out that, in general, an invertible ideal in a Prüfer ring need not be generated by regular elements: for such an example, see [AP].

$I = Ra + Rb$ for some $b \in I$. If $b \notin (\operatorname{rad} R)I$, we are done by taking $c = b$. Otherwise, Nakayama's Lemma implies that $I = Ra$, and we can just take $c = a$. $\square$

Using Lemma 2, we obtain the following result in generalization of [HL: (4.1)]. Note that, in contrast to [HL], R is *not* assumed to be a Prüfer ring in this result.

General Steinitz Isomorphism Theorem. *Let R be a ring with the $1\frac{1}{2}$ generator property having small 0-divisors. For any two invertible ideals I, J in R, we have an R-module isomorphism $I \oplus J \cong R \oplus IJ$.*

Proof. In general, any invertible ideal is regular (see [La$_2$: (2.17)]), so we can fix a regular element $x \in I$. Let $I_1 := xI^{-1} \subseteq I\,I^{-1} = R$. Then I_1 is an invertible ideal (with inverse $x^{-1}I$), and hence so is I_1J.

Case 1. $J \nsubseteq \operatorname{rad}(R)$. Since I_1 is invertible, this implies $I_1J \nsubseteq (\operatorname{rad} R)\, I_1$. Fix an element $a \in I_1J \setminus (\operatorname{rad} R)I_1$. Applying Lemma 2(2) to the invertible ideal I_1, we can write $I_1 = Ra + Rc$ for a suitable *regular* element $c \in I_1$. In particular, we have $I_1 = I_1J + Rc$. Multiplying this equation (in $Q(R)$) by $I_1^{-1} = x^{-1}I$, we get $R = J + x^{-1}c\,I$. Now $x^{-1}c\,I \cong I$ (as R-modules) since x and c are both regular. Thus,

$$I \oplus J \cong x^{-1}c\,I \oplus J \cong R \oplus x^{-1}c\,I\,J \cong R \oplus I\,J.$$

Here, the second isomorphism holds since the ideal $K := x^{-1}c\,I$ is comaximal with J. (The isomorphism is simply given by splitting the obvious epimorphism from $K \oplus J$ to $K + J = R$, the kernel of which is isomorphic to $K \cap J = K \cdot J$.)

Case 2. $J \subseteq \operatorname{rad}(R)$. Since J is invertible, there exist elements $q_j \in J^{-1}$ such that $\sum_j J\,q_j = R$. For a suitable index i, we have $J\,q_i \nsubseteq \operatorname{rad}(R)$. Write $q_i = s_i/s$, where $s_i \in R$, and s is a regular element in R. Arguing as in the proof of (2) $\Longrightarrow$ (4) in Lemma 1, we see that s_i must be regular, so q_i is a unit in $Q(R)$. By replacing J with the isomorphic R-ideal $J\,q_i \nsubseteq \operatorname{rad}(R)$, we are back to *Case 1.* $\square$

It is worthwhile to point out that the Steinitz Isomorphism Theorem does not hold in general for invertible ideals even in a Prüfer domain. For, if $I \oplus J \cong R \oplus IJ$ holds for all invertible ideals $I, J \subseteq R$, then, taking J to be isomorphic to I^{-1}, we get readily that $I \oplus I^{-1} \cong R \oplus R$, which would imply that I is 2-generated. But according to Schülting [Sch], there exists a (2-dimensional) Prüfer domain R with an (invertible) ideal that is 3-generated but not 2-generated. Thus, the Steinitz Isomorphism cannot hold for R.

§3. Decomposition of Finitely Generated Projective Modules

In this section, we study the structure of f.g. projective modules over Prüfer rings with small 0-divisors. The following result may be viewed as an analogue of the theorem of Cartan and Eilenberg [CE: Ch. I, Prop. 6.1] (see also [La$_2$: (2.29)]) on the decomposition of f.g. projective modules over semihereditary rings.

Theorem 3. *Let R be a Prüfer rings with small 0-divisors. Then any f.g. projective R-module P is isomorphic to a finite direct sum of invertible ideals in R.*

Proof. Note first that the second assumption on R implies that R is *connected.* In fact, if $e \in R$ is an idempotent $\neq 1$, then for $f = 1 - e \neq 0$, we have $fe = 0$, so $e \in \mathcal{Z}(R)$. Now $R = Re + Rf \Longrightarrow R = Rf$. Therefore, f is a unit, and so $f = 1$ and $e = 0$. The connectedness of R implies that *any f.g. projective R-module has a constant rank.*

We may assume in the following that $P \neq 0$. By a theorem of Bass [La$_1$: (24.7)], the radical $\mathrm{rad}(P)$ of P is a proper submodule, so there exists an element $a \in P \setminus \mathrm{rad}(P)$. Now choose a module Q such that $F := P \oplus Q$ is a free R-module. Since $\mathrm{rad}(F) = \mathrm{rad}(P) \oplus \mathrm{rad}(Q)$, we have

$$a \notin \mathrm{rad}(F) = \mathrm{rad}(R) \cdot F$$

by [La$_1$: (4.6)(3)]. Thus, if $a = \sum_i a_i e_i$ with respect to a basis $\{e_i\}$ of F, there exists an index j for which $a_j \notin \mathrm{rad}(R)$. Since $\mathcal{Z}(R) \subseteq \mathrm{rad}(R)$, it follows that a_j is a *regular* element in R.

Let $\pi_j : F \to R$ denote the j-th coordinate projection on F. Since P is f.g., $\pi_j(P)$ is a f.g. ideal containing a regular element a_j. Thus, $\pi_j(P)$ is an invertible ideal, and hence a projective R-module. This implies that the surjection $\pi_j : P \to \pi_j(P)$ splits, so P has a direct summand isomorphic to the invertible ideal $\pi_j(P)$. By invoking an inductive hypothesis on the rank of P, it follows that P is isomorphic to a finite direct sum of invertible ideals of R. $\square$

Remark. In the argument above, we used the f.g. assumption on P only in the third paragraph. If P is *not* f.g., the work in the second paragraph above can still be used as a beginning step for arguing that P contains a direct summand isomorphic to an invertible ideal. For this more sophisticated argument, see the proof of Theorem 9 below.

Note that the above theorem implies, in particular, that $\mathrm{Pic}(R)$ (the Picard group of R) is the same as $\mathrm{Cl}\,(R)$ (the ideal class group of R). This shows, incidentally, that the theorem may not hold if we do not assume that R has small 0-divisors. In fact, there exist examples of classical rings of quotients R whose Picard group is nontrivial (see, e.g. [La$_2$: (2.22)]); such rings are of course Prüfer rings with trivial ideal class groups.

We are now in a position to prove Theorem A stated in the Introduction. In the proof below, *R is assumed to be a Prüfer ring with the $1\frac{1}{2}$ generator property having small 0-divisors.*

Proof of Theorem A. (Sketch) For any f.g. projective R-module J of rank 1, let $\{J\}$ denote the isomorphism class of J, viewed as an element in the Picard group $\mathrm{Pic}(R)$. For any f.g. projective R-module P, let $[P] \in K_0(R)$, and let $\overline{[P]}$ denote its image in $\tilde{K}_0(R) = K_0(R)/\mathbb{Z}\cdot[R]$. We define a map $\alpha : \mathrm{Pic}(R) \to \tilde{K}_0(R)$ by $\alpha\{J\} = \overline{[J]}$. Note that, for any ideals $I, J \subseteq R$ with J invertible, the R-module J is (projective and hence) flat, so we have an R-module isomorphism $I \otimes_R J \cong IJ$. This and the General Steinitz Isomorphism Theorem readily imply that α is a group homomorphism. Using

Theorem 3, we see that α is surjective. Finally, the usual exterior algebra argument gives the injectivity of α. Thus, $\alpha : \mathrm{Pic}(R) \to \tilde{K}_0(R)$ is a group isomorphism, as desired. $\square$

Having completed the proof of Theorem A, we can easily derive the following corollary on the structure and classification of f.g. projective modules over the Prüfer rings in question. Its proof is essentially the same as that in the classical case of Dedekind domains, so we shall omit it.

Corollary 4. *Let R be a Prüfer ring with the $1\frac{1}{2}$ generator property having small 0-divisors. Then any f.g. projective R-module P of rank n is isomorphic to $R^{n-1} \oplus I$ where I is an invertible ideal, and the isomorphism class of P is determined by that of I and the rank n. In particular, f.g. projective R-modules satisfy the cancellation law, and f.g. stably free R-modules are free.*

§4. Rings with the Strong $1\frac{1}{2}$-Generator Property

In view of the results in the previous sections, it is of interest to find examples of rings satisfying the $1\frac{1}{2}$ generator property. As it turns out, it is actually easier to name some rings that satisfy a stronger property: let us say that a ring R has the *strong $1\frac{1}{2}$ generator property* if any invertible ideal $I \subseteq R$ is generated by two elements, the first of which can be any prescribed regular element of I. The property has been well studied for domains (see, for example, [Ka$_2$] and [FS][3]), and it is straightforward to generalize the notion to arbitrary rings as we did. To show that this is a good generalization, we first point out, in Propositions 5 and 6 below, that two known results in the literature extend nicely from domains to general (commutative) rings.

Proposition 5. (cf. Brewer-Klingler [BK]) *A ring R has the strong $1\frac{1}{2}$ generator property iff, for any invertible ideals I, J in R, we have an R-isomorphism $R/J \cong I/JI$.*

Proof. (Sketch) The "if" part follows by taking $J = Ra$, where a is any given regular element in I. (Note that I/Ra is a quotient of I/JI, which would be cyclic if $I/JI \cong R/J$.) For the "only if" part, consider any invertible ideals I, $J \subseteq R$. Since IJ is also invertible, there exists a regular element $a \in IJ$. Using the strong $1\frac{1}{2}$ generator property, we have $I = Ra + Rb$ for some $b \in I$. As in the proof of [BK: Prop. 1], one shows easily that multiplication by b induces a well-defined R-isomorphism from R/J to I/JI. $\square$

Proposition 6. (cf. Heitmann-Levy [HL]) *Any ring R of Krull dimension ≤ 1 has the strong $1\frac{1}{2}$ generator property.*

Proof. (Sketch) This was proved for Prüfer domains of Krull dimension 1 in [HL]. However, since we defined the (strong) $1\frac{1}{2}$ generator property to be a property on invertible ideals to begin with, the Prüfer assumption on R becomes unnecessary. The proof in [HL: p. 372] for the domain case can easily be adapted to *any* ring with Krull dimension

[3] The terminology we used here follows the one originally introduced by Heitmann and Levy [LS], but not that of Brewer-Klingler [BK] or of Fuchs and Salce [FS].

≤ 1. In fact, the only additional remark needed for the proof is that, if a is a regular element in such a ring R, then R/Ra, and hence any of its factor rings, has Krull dimension 0. This is true since no prime ideal containing Ra can be a minimal prime of R (according to [Ka$_4$: Th. 84]). $\square$

Of course, Prop. 6 generalizes the well known fact that Dedekind rings have the strong $1\frac{1}{2}$ generator property. In the following, we shall record several other conditions on a ring R, each of which will guarantee that R has the strong $1\frac{1}{2}$ generator property. *Any one of these conditions, coupled with the small* 0*-divisor property on a Prüfer ring, will thus give us enough assumptions to apply Theorem A.*

The sufficient conditions we state below for the strong $1\frac{1}{2}$-generator property are both already in the literature. However, the definitions we adopt in this paper are a bit different from those used in earlier work. For this reason, it will be prudent in recalling these known results to give a brief explanation for each.

Generalizing another definition introduced so far for domains (see, e.g. [FS: p. 97]), we say that a ring R has *finite character* if every *regular* element of R lies in at most finitely many maximal ideals of R; that is, for any regular element $a \in R$, R/Ra is a semilocal ring.

Proposition 7. (Gilmer-Heinzer [GH]) *Any ring R of finite character has the strong $1\frac{1}{2}$ generator property.*

(In fact, let $I \subseteq R$ be any invertible ideal, and let a be any given regular element in I. By assumption, a lies only in finitely many maximal ideals of R. Thus, by Theorem 3 of [GH], there exists $b \in I$ such that $I = Ra + Rb$.)

Proposition 8. (Griffin [Gr]) *Let R be a ring in which every regular ideal is invertible. Then R is a Prüfer ring of finite character (and hence R has the strong $1\frac{1}{2}$ generator property by Proposition 7).*

[The rings in questions are, of course, exactly those Prüfer rings R whose regular ideals are f.g. (or equivalently, satisfy the ACC). By Griffin's Theorem 17 in [Gr], any regular element in such a ring R lies in only finitely many prime ideals of R; in particular, R has finite character, and so Proposition 7 applies. Examples of (commutative) rings satisfying the hypothesis of Prop. 8 include: hereditary rings, local rings whose maximal ideals consist of 0-divisors, and classical rings of quotients (e.g. 0-dimensional rings, such as von Neumann regular rings or perfect rings).]

§5. Non Finitely Generated Projective Modules

In this section, we turn our attention to possibly non f.g. projective modules, and study the structure of such modules over a Prüfer ring R, assuming again that R has small 0-divisors. The goal of the section will be to prove Theorem B stated in the Introduction. We start by proving the first part of that theorem.

Theorem 9. *Let R be a Prüfer ring with small 0-divisors, and let P be any nonzero projective R-module. Then P has a direct summand isomorphic to an invertible ideal of R. In particular, P is indecomposable iff it is isomorphic to an invertible ideal.*

Proof. The proof here is a more sophisticated version of that of Theorem 3. The beginning step of the argument is still the second paragraph of that proof, which works for any nonzero projective module P. In that step, we showed that, starting with any element $a \in P \setminus \operatorname{rad}(P)$, there exists a linear functional $\pi_j : P \to R$ with $\pi_j(a)$ regular in R. Here, we can no longer say that the ideal $\pi_j(P)$ is f.g.; however, we can proceed alternatively as follows. Following Bass [Ba: §4], let

$$\begin{aligned} o_P(a) &= \{f(a) \in R : \; f \in P^* = \operatorname{Hom}_R(P, R)\}, \quad \text{and} \\ o'_P(a) &= \{p \in P : \; f(a) = 0 \;\Rightarrow\; f(p) = 0 \;\; \forall\, f \in P^*\}. \end{aligned}$$

By [Ba: Prop. 4.1], $o'_P(a) \cong o_P(a)^*$, and $o_P(a)$ is a f.g. ideal in R. By what we said above, $o_P(a)$ contains a regular element $\pi_j(a)$. Since R is a Prüfer ring, $o_P(a)$ is an invertible ideal, and hence a projective R-module. According to Bass [Ba: Prop. 4.1], this implies that $o'_P(a)$ is a direct summand of P. Since

$$o'_P(a) \cong o_P(a)^* \cong o_P(a)^{-1},$$

P has a direct summand isomorphic to an invertible ideal $I \cong o_P(a)^{-1}$. And, if P is indecomposable, then $P \cong I$. □

In the following, we shall write R^∞ for the countably infinite direct sum $R \oplus R \oplus \cdots$ (as an R-module). To use this module effectively, let us recall the famous Eilenberg-Mazur trick in the following special form.

Lemma 10. *For any ring R, we have $P \oplus R^\infty \cong R^\infty$ for any countably generated projective R-module P.*

Proof. For such a projective module P, there exists a surjection $\pi : R^\infty \to P$. Since π must split, we have $R^\infty \cong Q \oplus P$ for $Q = \ker(\pi)$. Thus, we have

$$\begin{aligned} P \oplus R^\infty &\cong P \oplus R \oplus R \oplus \cdots \\ &\cong P \oplus R^\infty \oplus R^\infty \oplus \cdots \\ &\cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \\ &\cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \\ &\cong R^\infty \oplus R^\infty \oplus \cdots \cong R^\infty, \end{aligned}$$

as desired. □

Lemma 11. *Let R be a Prüfer ring with the $1\frac{1}{2}$ generator property having small 0-divisors. If $P = P_1 \oplus P_2 \oplus \cdots$ where each P_i is a nonzero countably generated projective R-module, then P is free.*

Proof. By Theorem 9, we can write $P_i \cong I_i \oplus Q_i$, where I_i is an invertible ideal in R. By the General Steinitz Isomorphism Theorem in §2, we have $I_{2i-1} \oplus I_{2i} \cong R \oplus I_i'$, where $I_i' := I_{2i-1} I_{2i} \subseteq R$. Thus,

$$\begin{aligned} P &\cong (I_1 \oplus Q_1) \oplus (I_2 \oplus Q_2) \oplus \cdots \\ &\cong [(I_1 \oplus I_2) \oplus (I_3 \oplus I_4) \oplus \cdots] \oplus (Q_1 \oplus Q_2 \oplus \cdots) \\ &\cong [(R \oplus I_1') \oplus (R \oplus I_2') \oplus \cdots] \oplus (Q_1 \oplus Q_2 \oplus \cdots) \\ &\cong R^\infty \oplus P', \end{aligned}$$

where $P' := (I_1' \oplus I_2' \oplus \cdots) \oplus (Q_1 \oplus Q_2 \oplus \cdots)$. Since P' is a countably generated projective module, we conclude from Lemma 10 that $P \cong R^\infty \oplus P' \cong R^\infty$, as desired. □

We are now in a position to prove the rest of Theorem B.

Theorem 12. *Let R be as in Lemma 11. Then any infinite direct sum of nonzero countably generated projective R-modules is free, and any non countably generated projective R-module is free.*

Proof. Let $P = \bigoplus_i P_i$ where the P_i's are nonzero countably generated projective R-modules, and i ranges over some *infinite* indexing set Λ. Let Λ' be another copy of Λ. Since Λ' is infinite, we have

$$\mathrm{Card}(\mathbb{N} \times \Lambda') = \mathrm{Card}(\Lambda') = \mathrm{Card}(\Lambda).$$

Thus, after "identifying" Λ with $\mathbb{N} \times \Lambda'$, we can express the elements $i \in \Lambda$ in the form (n, i'), where $n \in \mathbb{N}$ and $i' \in \Lambda'$. We then have

$$P = \bigoplus_{i \in \Lambda} P_i = \bigoplus_{i' \in \Lambda'} P(i'), \tag{13}$$

where, for each $i' \in \Lambda'$, $P(i') := \bigoplus_{n \in \mathbb{N}} P_{(n,i')}$. By Lemma 11, each $P(i')$ is free, so by (13), P is also free. This proves the first part of the theorem.

For the second part, let P be any *non countably generated* projective R-module. By Kaplansky's theorem in [Ka$_3$], we can express P in the form $\bigoplus_{i \in \Lambda} P_i$ (for some indexing set Λ), where the P_i's are nonzero countably generated projective R-modules. Since P itself is not countably generated, Λ must be an infinite set. Thus, the first part of the theorem applies, showing that P is free. □

It seems plausible that, under the assumptions on R in Theorem 12, any countably but not finitely generated projective R-module P is also free. This would follow from Lemma 11 if we can decompose P as in that Lemma. However, we are not able to prove the existence of such a decomposition.

We close by recalling that most results in this note required the small 0-divisor assumption on R. The study of projective modules over general Prüfer rings (without the small 0-divisor assumption) awaits further effort.

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