D.6 Ordered Fields

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Definitions, Examples, and Early History

A field (or a division ring) F is said to be *ordered* if there is given a subset $P \subseteq F$ (called a positive cone) closed under addition and multiplication such that $P \cup (-P) = F$ and $P \cap (-P) = \{0\}$. Writing $a \geq b$ to mean $a - b \in P$, we get a total ordering of the elements of F. An ordered division ring must have characteristic 0; its positive cone $P = \{a : a \geq 0\}$ cannot contain -1, but must contain $\sum F^2$ (all sums of squares in F). Prototype examples are $\mathbb{Q}$ and $\mathbb{R}$ with their usual orderings, while fields such as $\mathbb{Q}(i)$ and $\mathbb{C}$ cannot be ordered. The ordering on $\mathbb{R}$ is *archimedean* in the sense that any element in the field is bounded by an element of $\mathbb{Q}$. Subfields of $\mathbb{R}$ with their inherited orderings turn out to be, in fact, *all* archimedean ordered division rings. A typical nonarchimedean ordered field is $\mathbb{R}(x)$, with $P \setminus \{0\}$ consisting of $f(x)/g(x)$ where $f, g \in \mathbb{R}[x]$ have positive leading coefficients. In this ordering, the variable t is "infinitely large" with respect to the subfield $\mathbb{R}$. By the same token, we can extend this ordering to one on $\mathbb{R}(x, y)$ in which y is infinitely large with respect to $\mathbb{R}(x)$.

Ordered division rings were first considered by Hilbert in his work on the foundations of geometry. Roughly speaking, an ordering corresponds to a "betweenness" notion for the points on a line in a geometry coordinatized by the division ring in question. Hilbert constructed the first example of a noncommutative (and hence nonarchimedean) ordered division ring, from which he obtained a geometry that satisfies Desargues' Theorem but not Pappus' Theorem. Historically, Hilbert's example also provided the first division ring that is infinite dimensional over its center.

Artin-Schreier Theory

In [1], Artin and Schreier defined a field to be *formally real* if $-1 \notin \sum F^2$, and showed that such fields are precisely the ones that can be ordered. Artin showed further that, if $\mathrm{char}(F) \neq 2$, then $a \in \sum F^2$ iff $a \geq 0$ w.r.t. all orderings of F. Using this result as one of his tools, Artin gave an affirmative answer to Hilbert's 17th Problem: for $F = \mathbb{R}(x_1, \ldots, x_n)$, $f \in \sum F^2$ iff $f(a_1, \ldots, a_n) \geq 0$ for all $(a_1, \ldots, a_n) \in \mathbb{R}^n$ where it is defined (and the same holds if $\mathbb{R}$ is replaced by $\mathbb{Q}$).

Central to the Artin-Schreier theory is the notion of a *real-closed field*: a formally real field F is real-closed if any proper algebraic extension of F is non-real. Such a field F captures the main algebraic features of the real field $\mathbb{R}$. There are several algebraic characterizations for such fields; for instance,

(1) F is formally real, $|F^*/F^{*2}| = 2$, and every odd-degree polynomial has a root in F.

(2) F has finite codimension > 1 in its algebraic closure.

(3) $F(\sqrt{-1}) \supsetneq F$ is an algebraically closed field.

Real-closed fields are ubiquitous as every ordered field (F, P) has a unique "real-closure", that is, a real-closed algebraic extension F_P whose unique ordering extends P.

Real Valuations and Real Places

A (Krull) valuation v on a field F is said to be *compatible* with an ordering P if its valuation ring A is convex w.r.t. P. (Alternatively, we can also say that A is compatible with P.) In this case, P "pushes down" to an ordering on the residue class field of v. The principal example is the following. If E is any subfield of F, the convex hull A of E w.r.t. P is the smallest valuation ring $\supseteq E$ compatible with P, and its maximal ideal $\mathfrak{m}$ is the set of elements infinitely small w.r.t. E. Here, P pushes down to an ordering of $A/\mathfrak{m}$ with respect to which $A/\mathfrak{m}$ is archimedean over $\bar{E}$ (the image of E). The value group Γ of the valuation is the ordered group of "E-archimedean classes" on F. Taking $E = \mathbb{Q}$, we get in particular the *Hahn valuation* of the ordered field (F, P), together with a natural *Hahn embedding* of (F, P) into the ordered Laurent series field $\mathbb{R}((\Gamma))$.

Given any valuation v on F with residue field k and value group Γ, the work of Baer and Krull provides a description of the set of all orderings on F compatible with v, in terms of the orderings on k and the characters on the group Γ/Γ^2.

Theorems on real valuations are often related to those on places into real-closed fields. The systematic study of such real places began with S. Lang, who proved that, for any formally real function field F over a real-closed constant field k, there exists a k-place $\lambda : F \to k \cup \{\infty\}$, and λ can be chosen to be finite on a finite number of prescribed points in F. An important consequence of this is Lang's Homomorphism Theorem: *if A is a commutative finitely generated k-domain with quotient field F, then there exists a k-algebra homomorphism from A to k.* These results constitute the beginning of a theory of real function fields.

Tarski's Transfer Principle and Robinson's Nonstandard Reals

The algebraic theory of ordered fields has played a very substantial role in the creation of the model theory of fields in logic. In 1948, Tarski showed that the theory of real-closed ordered fields admits quantifier elimination in the language of ordered rings (every "formula" is equivalent to a quantifier-free formula). From this, it follows that the theory of real-closed fields is "model-complete", that is, if $E \subseteq F$ are real-closed fields, then an elementary sentence about ordered fields with parameters from E holds in $(F, \leq)$ iff it holds in $(E, \leq)$. A more popular form of these results is *Tarski's Transfer Principle*, to the effect that every elementary sentence about ordered fields which holds in $(\mathbb{R}, \leq)$ also holds in every real-closed field $(F, \leq)$. (This would quickly imply, for instance, an affirmative answer to Hilbert's 17th Problem.) These famous results of Tarski set an important paradigm for much of the subsequent model-theoretic research in logic.

Another possibly surprising development is A. Robinson's invention of nonstandard analysis in the 1960s. By using an ultrapower of the real number system $\mathbb{R}$ (w.r.t. a non-principal ultrafilter on $\mathbb{N}$), Robinson constructed a "nonstandard" number domain ${}^*\mathbb{R} \supsetneq \mathbb{R}$, which is real-closed, nonarchimedean, and such that every real function $y = f(x)$ extends canonically to a function on ${}^*\mathbb{R}$. In the nonstandard reals ${}^*\mathbb{R}$, infinitesimals exist, and can be used to efficiently develop the differential and integral calculus on $\mathbb{R}$, much in the way Leibniz originally did in the 17th century, but now with fully justified proofs!

Orderings and Quadratic Forms

For a field F of characteristic $\neq 2$, the Witt ring $W(F)$ consists of "Witt classes" of nonsingular quadratic forms over F. (Two such forms belong to the same Witt class if their anisotropic parts are isometric.) The $+$ and $\times$ operations on $W(F)$ are given by the $\perp$-*sum* and *tensor product* of quadratic forms. (For instance, for the unary forms ax^2 and by^2 $(a, b \in F^*)$, the $\perp$-sum is $ax^2 + by^2$, and the tensor product is $ab\,z^2$.) The nicest nontrivial example of a Witt ring is that for a real-closed field K: in this case, the *signature map* (assigning to any quadratic form its signature) gives a ring isomorphism $W(K) \cong \mathbb{Z}$, according to Sylvester's Law of Inertia. More generally, if (F, P) is any ordered field, taking signatures defines a ring surjection $W(F) \to \mathbb{Z}$. This is essentially the functorial map $W(F) \to W(F_P)$, if we identify $W(F_P)$ with $\mathbb{Z}$.

For any formally real field F, the set of orderings X_F carries the *Harrison topology,* with a subbasis given by $\{P : a \in P\}$ where a ranges over F. This makes X_F into a Boolean (compact, Hausdorff, totally disconnected) space. By taking the "total signature", we get a ring homomorphism σ from $W(F)$ to $\mathcal{C}(X_F, \mathbb{Z})$, the ring of continuous functions from X_F to $\mathbb{Z}$ (the latter with the discrete topology). According to *Pfister's Local-Global Principle,* the kernel of σ is precisely $W(F)_t$, the ideal of torsion elements in $W(F)$. Pfister also showed that $W(F)_t$ is a 2-primary group. In case F is nonreal, all of this remains formally true as $W(F)$ itself is 2-primary in that case. In fact, if $-1 = a_1^2 + \cdots + a_s^2$ in F with s chosen minimal, then the "level" s of the nonreal field F has the form 2^n for some n (by another theorem of Pfister), and one has $2^{n+1}\,W(F) = 0$.

The quotient ring $W(F)/W(F)_t \cong \mathrm{im}(\sigma) \subseteq \mathcal{C}(X_F\,\mathbb{Z})$ is called the *reduced Witt ring* of F. More generally, if T is any *preordering,* that is, an intersection of orderings, one can construct a reduced Witt ring $W_T(F)$, whose elements have the form $\langle a_1, \ldots, a_n\rangle$, where two such forms are identified if they have the same dimension n, and the same signature with respect to all orderings containing T. We get back the classical reduced Witt ring $W(F)/W(F)_t$ by letting $T = \sum F^2$ in case F is formally real.

The consideration of quadratic forms has also led to the study of many special kinds of formally real fields, such as euclidean fields, SAP fields (fields with a Strong Approximation Property), pythagorean and superpythagorean fields (and some of their hereditary versions), etc. For more detailed information on these (and on other material surveyed in this article), see the references [2]-[5] below.

Generalizations

The theory of ordered fields sketched above has been generalized in many directions. E. Becker initiated a theory of orderings of higher level: in this theory, a positive cone P satisfies only the weaker condition that F^*/P^* is a cyclic group of even order (instead of order 2). A large part of the Artin-Schreier theory and Krull-Baer theory generalizes to this setting, and the theory can be used to study the structure of sums of even powers in a field. Other generalizations include the theory of preorderings on fields, formally p-adic fields, ordered and lattice-ordered rings (see C45), $*$-ordered division rings, and the theory of real spectra of commutative rings with its various applications to Nullstellensätze and real algebraic geometry.

References

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