# C.13 Orders and Maximal Orders

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### Definitions and Examples

Loosely speaking, the theory of orders and maximal orders enables us to study certain rings and their representations in an integral, noncommutative setting, in generalization of the arithmetic study of number rings.

Let $R$ be a commutative domain, with quotient field $K$, and let $A$ be a finite-dimensional $K$-algebra. By a *full $R$-lattice* in $A$, we mean a finitely generated $R$-submodule $\Lambda \subseteq A$ such that $K\Lambda = A$. If such a $\Lambda$ happens to be a subring of $A$, we say that $\Lambda$ is an *$R$-order*. If, moreover, $\Lambda$ is not properly contained in any $R$-order of $A$, we say it is a *maximal order* (with $R$ understood). For instance, if $M \subseteq A$ is any full $R$-lattice, then $O_\ell(M) = \{a \in A : aM \subseteq M\}$ is an $R$-order, called the *left order* of $M$ (and the *right order* $O_r(M)$ is defined similarly). If $a_1, \ldots, a_n \in K$ are integral over $R$, then $R[a_1, \ldots, a_n]$ is an $R$-order in $K$.

Here are some more examples. Let $\tau$ be the golden ratiio $(1+\sqrt{5})/2$. For any positive integer $n$, $\Lambda_n = \mathbb{Z} + \mathbb{Z}\cdot n\sqrt{5}$ is a $\mathbb{Z}$-order in $\mathbb{Q}(\sqrt{5})$, and $\mathbb{Z}+\mathbb{Z}\cdot\tau$ is the unique maximal $\mathbb{Z}$-order. More generally, for any number field $A$, the full ring of algebraic integers $\Lambda$ is the unique maximal $\mathbb{Z}$-order. For some *noncommutative* examples, take $A = \mathbb{M}_2(\mathbb{Q})$. For any integer $n > 0$, $\{(a_{ij}) \in \mathbb{M}_2(\mathbb{Z}) : n|a_{21}\}$ is a $\mathbb{Z}$-order in $\mathbb{M}_2(\mathbb{Q})$, and it is contained in the maximal $\mathbb{Z}$-order $\mathbb{M}_2(\mathbb{Z})$. For any finite group $G$, the group ring $\Lambda = \mathbb{Z}G$ is a $\mathbb{Z}$-order in $\mathbb{Q}G$. In the case where $G$ is abelian, Maschke's Theorem gives $\mathbb{Q}G = A_1 \times \cdots \times A_r$, where the $A_i$'s are suitable number fields. Here, $\Lambda$ is contained in the unique maximal order $\Lambda_1 \times \cdots \times \Lambda_r$, where $\Lambda_i$ is the full ring of algebraic integers in $A_i$. In the $\mathbb{Q}$-algebra of quaternions $\mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$, the $\mathbb{Z}$-span of $1, i, j, k$ is a $\mathbb{Z}$-order, and so is the $\mathbb{Z}$-span of $1, 2i, 2j$ and $k$, while the $\mathbb{Z}$-span of $i, j, k$ and $(1+i+j+k)/2$ (the "Hurwitz ring of integral quaternions") is a maximal $\mathbb{Z}$-order.

The term "order" used in this article is not to be confused with the same term used in the sense of an "ordering" (as, for instance, in C.49 and D.6). This confusion is unfortunate, but is also rather difficult to avoid. The word "order" as defined in the second paragraph above is a direct translation of the German word "Ordnung", which was commonly used in the 19th century number theory literature to refer to rings of algebraic integers (see, e.g. [1]). The extension of the notion of "Ordnung" to a *noncommutative* setting seemed to have stemmed from the work of Emmy Noether.

### Orders in Separable Algebras

For $R$, $K$, $A$ as above, we say $A$ is a *separable $K$-algebra* if $A$ is semisimple and the center of each simple component of $A$ is a separable field extension of $K$. For such (and only such) $A$, $A \otimes_K E$ remains a semisimple $E$-algebra for any field $E \supseteq K$. Assuming henceforth that $R$ is noetherian and integrally closed, *any $R$-order in $A$ is contained in a maximal order*, and each maximal order arises as a direct product of maximal $R$-orders in the simple components of $A$.

Maximal orders behave well with respect to localizations and completions. First, an $R$-order $\Lambda$ is maximal iff, for any maximal ideal $\mathfrak{p} \subset R$, $\Lambda_{\mathfrak{p}}$ is a maximal $R_{\mathfrak{p}}$-order in $A$. And, if $R$ is already local, with maximal ideal $\mathfrak{m}$ and $\mathfrak{m}$-adic completion $\hat{R}$, then an $R$-order $\Lambda \subseteq A$ is maximal iff its completion $\hat{\Lambda} = \Lambda \otimes_R \hat{R}$ is a maximal $\hat{R}$-order in $\hat{A}$.

### Maximal Orders over Dedekind Rings

*From now on, we assume $R$ is a Dedekind ring (and $A$ is a separable $K$-algebra).* The consideration of maximal $R$-orders in $A$ is substantially reduced to the case where $R$ is a complete DVR (discrete valuation ring). We first consider the key case where $A$ is a division $K$-algebra $D$. Here, there is a *unique* maximal order, namely the integral closure of $R$ in $D$. This $\Lambda$ is a "noncommutative DVR" in that it has an element $\pi$ (called a "uniformizer") such that the 1-sided and 2-sided nonzero ideals of $\Lambda$ are all of the form $\pi^i\Lambda$. If $\pi_0$ is the uniformizer of $R$, then $\pi_0\Lambda = \pi^e\Lambda$, where $e$ is the *ramification index* of $D$ over $K$. With the *inertial degree* $f$ defined to be $\dim_{R/\pi_0 R} \Lambda/\pi\Lambda$, we have $ef = [D : K]$, exactly as in the case of a field extension of the complete field $K$. If, moreover, $R/\pi_0 R$ is a finite field and the center of $D$ is exactly $K$, then in fact $e = f$, and $D$ is a "cyclic algebra" completely determined by its index $e$ together with another integer $r \in (0, e)$ relatively prime to $e$. (The image of $r/e$ in $\mathbb{Q}/\mathbb{Z}$ is the *Hasse invariant* of $[D]$ in the Brauer group of $K$, and all elements in $\mathbb{Q}/\mathbb{Z}$ are realized this way). More generally, if $A = \mathbb{M}_n(D)$, all maximal orders in $A$ are conjugate to $\mathbb{M}_n(\Lambda)$. In view of Wedderburn's Theorem, this completes the determination of maximal orders over a complete DVR.

Continuing with the case $R$ is a DVR (but not necessarily complete), a maximal order $\Lambda$ in a central simple $K$-algebra $A$ is always a principal ideal ring (all 1-sided ideals are principal), and $\Lambda/\mathrm{rad}(\Lambda)$ is a simple artinian ring. Finally, in the general case when $R$ is a Dedekind ring, any maximal order $\Lambda$ in a separable $K$-algebra $A$ is a hereditary ring, in the sense that any left or right ideal of $\Lambda$ is projective as a $\Lambda$-module. In the case when $A$ is simple, that is, $A = \mathbb{M}_n(D)$ where $D$ is a division $K$-algebra, any maximal order in $A$ is Morita equivalent to one in $D$.

The ideal structure for a maximal $R$-order $\Lambda$ in a separable $K$-algebra $A$ bears a good resemblance to that in the Dedekind ring $R$ itself. By the word "ideal", we mean here a 2-sided ideal in $\Lambda$ that is a full lattice. With this understanding, prime ideals in $\Lambda$ are maximal, the product of prime ideals is commutative, and any ideal is uniquely a product of prime ideals. For any ideal $I$, we also have $I \cdot I^{-1} = I^{-1} \cdot I = \Lambda$, and $(I^{-1})^{-1} = I$, where $I^{-1}$ means $\{x \in A : IxI \subseteq I\}$, just as for nonzero ideals in Dedekind rings. In the case when $A$ is simple, there is a natural one-one correspondence (defined by contraction) between the prime ideals of $\Lambda$ and the (nonzero) prime ideals of $R$.

There is also a theory of one-sided ideals in $A$, but it is more technical. A full $R$-lattice in $A$ is called a *normal ideal* if its left (or equivalently, right) order in $A$ is a maximal order. Certain normal ideals can be multiplied, and with respect to such a "proper" multiplication, all normal ideals form a groupoid, called the *Brandt groupoid* of $A$.

## Lattices over Orders

By a left $\Lambda$-*lattice*, we mean a left $\Lambda$-module that is finitely generated and torsion-free as an $R$-module. For instance, for a finite group $G$, the study of $RG$-lattices is basically the theory of integral representations of groups.

Certainly, every $\Lambda$-lattice breaks up into a direct sum of indecomposable ones. However, the isomorphism types of the intervening indecomposable lattices are not uniquely determined, as is already witnessed by the case $\Lambda = R$. We express this by saying that, for lattices, the Krull-Schmidt Theorem (generally) fails. Nevertheless, there are some important cases in which the KS Theorem holds; e.g. it does when

(1) $R$ is a *complete* DVR, or

(2) $R$ is a DVR and $A$ is a split $K$-algebra (direct product of matrix rings over $K$).

Independently of whether the KS Theorem holds, it is of interest to determine the number $n(\Lambda)$ of indecomposable $\Lambda$-lattices (up to isomorphism). To demonstrate the subtlety of this problem, we just mention a sample result of A. Jones. For a group ring $\Lambda = \mathbb{Z}G$, $n(\Lambda)$ is finite iff for any prime $p$ dividing $|G|$, a $p$-Sylow group of $G$ is cyclic of order $p$ or $p^2$.

An important tool used in classifying lattices is the notion of *genera*: two $\Lambda$-lattices $M$, $N$ are said to be *in the same genus* (written $M \vee N$) if $M_\mathfrak{p} \cong N_\mathfrak{p}$ as $\Lambda_\mathfrak{p}$-lattices for every maximal ideal $\mathfrak{p} \subset R$. The latter condition can also be checked over the completions $\hat{R}_\mathfrak{p}$. This remark implies for instance that, if $M \oplus L \cong N \oplus L$ for some lattice $L$, or if $M^r \cong N^r$ for some integer $r$, then $M \vee N$. In case $\Lambda$ is a *maximal* order, $M \vee N$ amounts to $KM \cong KN$ as $A$-modules. Another interesting problem is to compute the number of lattices in the genus of a given lattice $M$.

The remarks in the paragraph above linked the study of genera to questions on *cancellation* of lattices. Over arbitrary orders, however, we do not expect cancellation to hold. In fact, examples of Swan showed that cancellation can fail already for projective lattices over maximal orders.

Assume now that $K$ is a global field (that is, either a number field or a function field in one variable over a finite field). A well-known result in number theory says that the Dedekind ring $R$ has a finite class number, i.e. there are only finitely many $R$-isomorphism types of ideals. This famous result admits a strong generalization to $\Lambda$-lattices. According to the *Jordan-Zassenhaus Theorem,* if $\Lambda$ is any $R$-order in a semisimple $K$-algebra $A$, then for any positive integer $n$, there are only finitely many isomorphism types of left $\Lambda$-lattices of $R$-rank $\leq n$. For instance, in the special case $n = \dim_R \Lambda$, there are only finitely many isomorphism types of full left $\Lambda$-lattices in $A$. (These are known as "left $\Lambda$-ideals".) For detailed expositions (and a survey) on the general theory of lattices, see [2]-[4].

## Things We Have No Space For

There are many other directions in the research on orders that we have not touched upon. Let us close by mentioning just a few of the most fruitful directions.

(1) *Unit groups of orders* (for instance $\mathrm{GL}_n(\mathbb{Z})$ in $\mathbb{M}_n(\mathbb{Z})$, or units in quaternionic $\mathbb{Z}$-orders) are basic examples of arithmetic groups. These groups are finitely presented — in generalization of Dirichlet's Unit Theorem for rings of algebraic integers. Units in group rings already constitute a big topic for research, and require techniques specific to the group-theoretic setting.

(2) *$K$-theory of orders* is the study of Grothendieck groups and Whitehead groups of orders, and their higher $K$-theoretic analogues. The computations of class groups and Picard groups of group rings, Milnor's $K_2$-groups of rings of algebraic integers, etc. are special cases. These are all finitely generated abelian groups. (For general background on the $K$-theory groups, see H7 and H8.)

(3) *A plethora of orders*: Besides maximal orders, many other kinds of orders have been investigated; e.g. clean orders, hereditary orders, Gorenstein orders, and Bass orders, etc. The main job is to "classify" each such kind of orders, to the extent possible.

(4) *Analytic methods*: For any $\mathbb{Z}$-order $\Lambda$, L. Solomon has defined the zeta function: $\zeta_\Lambda(s) = \sum_M [\Lambda : M]^{-s}$, where $M$ ranges over the (full) left ideals in $\Lambda$. When $\Lambda$ is a full ring of algebraic integers, this is Dedekind's zeta function, which in turn generalizes Riemann's zeta function. It is hoped that the study of $\zeta_\Lambda$ will be useful for noncommutative orders in the same way Dedekind's zeta function is useful for classical number theory.

## References

[1] R. Dedekind: *Über die Anzahl der Ideal-Klassen in den verschiedenen Ordnungen eines endlichen Körpers (1877),* Gesammelte Mathematische Werke, Erster Band, pp. 105-158, Braunschweig, 1930.

[2] I. Reiner: *Maximal Orders,* Academic Press, 1975.

[3] I. Reiner: *A survey of integral representation theory,* Bull. AMS **76**(1970), 159-227.

[4] R. Swan and E. G. Evans: *K-Theory of Finite Groups and Orders,* Lecture Notes in Math. Vol. 149, Springer-Verlag, Berlin-Heidelberg-New York, 1970.