C.20 Local Rings

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Examples, History, and Conventions

Consider the ring $A = \{a/b : a, b \in \mathbb{Z}, b \text{ odd}\}$. In A, all odd primes are units, and $2A$ is the only maximal ideal. The ring $B = k[[x, y]]$ of formal power series in x, y with coefficients in a field k has a similar feature: any power series outside of $xB + yB$ can be formally inverted, so $xB + yB$ is the only maximal ideal of B. Yet another "similar" ring is $C = \mathbb{Z}/p^n\mathbb{Z}$ for any prime p, except that C is no longer a domain when $n > 1$.

A commutative ring R is called a *local ring* if it has exactly one maximal ideal, or equivalently, $R \neq 0$ and the nonunits of R form an ideal $\mathfrak{m}$. For such a local ring $(R, \mathfrak{m})$, $R/\mathfrak{m}$ is a field, called the *residue field* of R. The rings A, B, C in the last paragraph are good prototype examples.

Commutative local rings were introduced by Krull [5], who called them "Stellenringe". In [2], Chevalley coined the English translation "local rings", and further developed their theory. In both [5] and [2], local rings were taken to be *noetherian*: a cause later championed by Bourbaki. Here, however, we follow Kaplansky, Atiyah-Macdonald, Matsumura [6], Bruns-Herzog [1], and define local rings *without* the noetherian assumption; in Bourbaki's terminology, these are the "quasi-local" rings.

In the following, we give a quick survey on local rings (referring the reader to [1] and [6] for detailed expositions). Until further notice, *all rings are assumed to be commutative.*

Sources of Local Rings, and Algebraic Geometry

Local rings are important because they are ubiquitous. First, all fields are local rings, although some of us would say that these shouldn't count. Any indecomposable artinian ring is (noetherian) local, and if R is a local ring, then so is any nonzero factor ring of R. Also, if $\mathfrak{m}$ is a maximal ideal of any ring R, then $R/\mathfrak{m}^n$ is a local ring for any $n \geq 1$. Lastly, a ring R contained in a field K is called a *valuation ring* of K if every $a \in K \backslash R$ has $a^{-1} \in R$. These rings arise from the theory of valuations and places, and are always local (though they are usually not noetherian). Krull was certainly deeply interested in these non-noetherian local rings as well.

If $\mathfrak{p}$ is a prime ideal in any ring R, then equivalence classes of the fractions $\{r/s : r, s \in R,\ s \notin \mathfrak{p}\}$ form a local ring with maximal ideal $\{p/s : p \in \mathfrak{p},\ s \notin \mathfrak{p}\}$. This ring, denoted by $R_\mathfrak{p}$, is called the *localization* of R at the prime $\mathfrak{p}$. By studying the local ring $R_\mathfrak{p}$, we hope to get insight on the prime $\mathfrak{p}$. We can localize R-modules too by tensoring them with $R_\mathfrak{p}$; this is known to preserve exact sequences. By localization with respect to *all* primes $\mathfrak{p}$, we get a powerful "local-global method" for studying R and its modules.

Local rings are very much "geometric" objects; they arise naturally in algebraic (resp. analytic) geometry as rings of "germs" of functions at a point of an algebraic (resp. analytic) variety. For instance, the local ring at the origin for the twisted cubic $x^2 = y^3$ is the localization of its coordinate ring $\mathbb{C}[x, y]/(x^2 - y^3)$ at the prime ideal $(\bar{x}, \bar{y})$. Classically, purely algebraic results on local rings were obtained with the goal of studying the local behavior of algebraic varieties. The theory of local rings developed by Krull, Chevalley, Zariski, Cohen, and others went a long way towards the rewriting of the algebraic foundations of classical algebraic geometry in the 1940s. Later, in the hands of Grothendieck, varieties were further generalized into "schemes". Each scheme X comes with a structural sheaf $\mathcal{O}_X$, whose "stalk" at each point $x \in X$ is a local ring $\mathcal{O}_{X,x}$.

Toolkit for Local Algebra

Some principal theorems on local rings $(R, \mathfrak{m})$ are as follows:

C.20.1. Krull's Intersection Theorem. If R is noetherian, then $\bigcap_{i=1}^{\infty} \mathfrak{m}^i = 0$.

C.20.2. Krull's Dimension Theorem. If R is noetherian, it has finite Krull dimension n (the length of a longest chain of prime ideals). There exist $x_1, \ldots, x_n \in \mathfrak{m}$ which generate an ideal with radical $\mathfrak{m}$: this is called a *system of parameters* for R.

C.20.3. Nakayama's Lemma. For any finitely generated R-module M, $\mathfrak{m} M = M$ implies that $M = 0$.

C.20.4. Kaplansky's Theorem. Any projective module over R is free.

Any noetherian local ring $(R, \mathfrak{m})$ comes with a Hausdorff topology whose system of neighborhoods at 0 is given by the powers of $\mathfrak{m}$. With respect to this topology, R has a completion $(\hat{R}, \hat{\mathfrak{m}})$, which is also a noetherian local ring (of the same Krull dimension). Going from R to $\hat{R}$ (and vice versa) is often a very effective tool in local algebra.

Every finitely generated module $M \neq 0$ over a noetherian local ring $(R, \mathfrak{m})$ comes with many invariants, such as $\dim(M)$ (Krull dimension of $R/\mathrm{ann}(M)$), the projective dimension $\mathrm{pd}_R(M)$ and the injective dimensions $\mathrm{id}_R(M)$ (the lengths of the shortest projective and injective resolutions of M), and $\mathrm{depth}(\mathfrak{q}, M)$ (length of a maximal M-regular sequence in an ideal $\mathfrak{q} \subseteq \mathfrak{m}$). In case $\mathrm{pd}_R(M) < \infty$, one has the following depth formula of Auslander and Buchsbaum:

$$\mathrm{pd}_R(M) = \mathrm{depth}(\mathfrak{m}, R) - \mathrm{depth}(\mathfrak{m}, M).$$

In case $M/\mathfrak{q}M$ has finite length over R, there exists a *Hilbert-Samuel polynomial* $P_{\mathfrak{q}}(t, M) \in \mathbb{Q}[t]$ of degree $d = \dim(M)$ such that, for large n, $P_{\mathfrak{q}}(n, M) = \ell_R(M/\mathfrak{q}^{n+1}M)$. (For an explicit example of this, see the next section.) The leading term of $P_{\mathfrak{q}}(t, M)$ has the form $e_{\mathfrak{q}}(M)\,t^d/d!$, where $e_{\mathfrak{q}}(M)$ is a positive integer called the *multiplicity* of M with respect to $\mathfrak{q}$. Under suitable assumptions, $e_{\mathfrak{q}}(M)$ is given by the Euler-Poincaré characteristic of the homology of a Koszul complex. In the case $M = R/\mathfrak{p}$ where $\mathfrak{p} + \mathfrak{q}$ is $\mathfrak{m}$-primary, this enabled Serre to express the intersection multiplicity of two subvarieties at a point of a given variety in terms of lengths of certain Tor-modules.

Regular Local Rings and Their Generalizations

A noetherian local ring $(R, \mathfrak{m})$ of dimension n is said to be *regular* if the maximal ideal $\mathfrak{m}$ can be generated by n elements. Two further characterizations are as follows:

(1) $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = n$; and

(2) the "associated graded ring" $\bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is isomorphic to the graded polynomial ring $k[x_1, \ldots, x_n]$, where $k = R/\mathfrak{m}$.

The characterization (2) is extremely powerful. For instance, it implies that R must be an integral domain, and that the Hilbert-Samuel polynomial $P_{\mathfrak{m}}(t, R)$ (for the free module R with respect to the maximal ideal $\mathfrak{m}$) has the very simple form $\binom{t+n}{n}$. Regular local rings were utilized effectively by Zariski to give a rigorous theory of simple points on an algebraic variety.

In general, if $(R, \mathfrak{m})$ is any regular local ring of dimension n, then so is its completion $(\hat{R}, \hat{\mathfrak{m}})$. For instance, the formal power series ring $k[[x_1, \ldots, x_n]]$ over any field k is a complete regular local ring of dimension n. The structure of complete regular local rings was determined by I.S. Cohen. In 1946, Cohen [3] showed how a general complete regular local ring is related to formal power series rings over fields and discrete valuation rings.

Two spectacular results on regular local rings were obtained in the 1950s. First, Serre and Auslander-Buchsbaum showed that regular local rings are precisely the noetherian local rings of finite global dimension (and indeed global dimension agrees with Krull dimension). This implies, for instance, Krull's conjecture that prime localizations of regular local rings remain regular. Second, Auslander and Buchsbaum proved that regular local rings are unique factorization domains, which is in part motivated by Weierstrass' Theorem that power series rings $k[[x_1, \ldots, x_n]]$ (over a field k) are UFDs.

One has the following hierarchy for noetherian local rings:

$$\text{regular} \Longrightarrow \text{complete intersection} \Longrightarrow \text{Gorenstein} \Longrightarrow \text{Cohen-Macaulay},$$

which gives three interesting and increasingly broader classes of generalizations of regular local rings. "Complete intersection" means a factor ring of a regular local ring by an ideal generated by a regular sequence. "Gorenstein" means a noetherian local ring R with finite injective dimension over itself ($\mathrm{id}_R(R) < \infty$). "Cohen-Macaulay" means a noetherian local ring $(R, \mathfrak{m})$ of dimension n such that $\mathfrak{m}$ has a regular sequence of length n (see C.46). Prompted by Macaulay's classical work on ideals in polynomial rings, Cohen-Macaulay rings have now found deep applications in invariant theory, algebraic geometry, polytope geometry, and algebraic combinatorics. Most remarkably, Mark Haiman's recent solution (in [4]) of the $n!$ Conjecture and Macdonald's Positivity Conjecture in symmetric function theory rests on proving that a certain isospectral Hilbert scheme is Gorenstein and Cohen-Macaulay.

Noncommutative Local Rings

In this section, we waive the commutativity assumption on our rings. Local rings are important in noncommutative ring theory as well. An arbitrary ring R is called *local* if it has a unique maximal left ideal $\mathfrak{m}$. We see easily that $\mathfrak{m}$ is also the unique maximal right ideal of R, and that $R\backslash\mathfrak{m}$ is exactly the group of units of R. Here we get a *residue division ring* $R/\mathfrak{m}$. Some examples of noncommutative local rings are: (1) division rings; (2) group rings kG for k a field of characteristic p and G any p-primary group; (3) $n \times n$ upper triangular matrices with constant diagonals over any division ring; and (4) the endomorphism ring of any indecomposable module of finite length (over any ring).

Various properties of commutative local rings still hold: for instance, the facts (C.20.3) and (C.20.4) do not depend on commutativity. One of the most important theorems making use of noncommutative local rings is the following

C.20.5. Krull-Schmidt-Azumaya Theorem. *Let $\{M_i, N_j\}$ be indecomposable modules over any ring A such that $\bigoplus_{i\in I} M_i \cong \bigoplus_{j\in J} N_j$. If the endomorphism rings $\mathrm{End}_A(M_i)$ are all local, then there is a bijection $\pi : I \to J$ such that $M_i \cong N_{\pi(i)}$ for all $i \in I$.*

Some attempts have been made toward developing a theory of regular local rings in the noncommutative setting, but the situation here is more complicated. It seems reasonable to define "regularity" of a local ring R by stipulating that R be 2-sided noetherian, with finite left and right global dimension (which must then be equal). *Is R then a domain*? This is known to be true if the global dimension is ≤ 3 (results of M. Ramras and R. Snider), but has remained unknown in general. On the other hand, there are other notions of noncommutative regularity (e.g. prompted by the consideration of graded rings), which we will not have space to explore here.

References

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