# On the Linkage of Quaternion Algebras

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For quaternion division algebras $B$, $C$ over a field $F$ (of any characteristic), a well-known theorem of Albert [1] and Sah [6] states that the following conditions are equivalent:

(1) $B \otimes_F C$ *is not a division algebra;*
(2) $B$ *and* $C$ *have a common quadratic splitting field;*
(3) *some quadratic field extension of* $F$ *can be embedded (over* $F$*) in both* $B$ *and* $C$.

In the case where $F$ has characteristic 2, there is a further refinement of this theorem, due to Draxl, which states that the above conditions are also equivalent to[1]:

(4) $B$ *and* $C$ *have a common separable quadratic splitting field;*
(5) *some quadratic separable field extension of* $F$ *can be embedded in both* $B$ *and* $C$.

Draxl's original proof in [3] was not easy (for me) to follow. Subsequent proofs of the equivalence of (1)–(5) using more advanced tools (respectively, algebraic geometry and the theory of Clifford algebras) appeared in this Bulletin in Tits [7] and Knus [5]. While teaching a course in the theory of division rings, I stumbled upon a short and completely elementary proof of Draxl's part of the above theorems. This proof is recorded below in order to make Draxl's result more easily accessible to non-experts. It has also been known for some time that (4) and (5) are no longer equivalent to (1)–(3) if the word "separable" is replaced by "inseparable". This will be demonstrated as well by an example that is simpler and easier to verify than the ones in Baeza [2: p. 134] and Knus [5: pp. 335-336].

*Throughout the following, we assume* $\text{char}(F) = 2$. Recall that, for $a \in F$ and $b \in F^*$, $[a,\, b)_F$ denotes the $F$-quaternion algebra generated by $i$, $j$ with the relations $i^2 + i + a = 0$, $j^2 = b$, and $ij = j(i+1)$. The following three standard isomorphisms for quaternion algebras will be useful:

(6) $[a,\, b)_F \cong [a+b,\, b)_F$;
(7) $[a,\, b)_F \cong [a + x^2 + x,\, b)_F$ *for any* $x \in F$. *In particular,* $[x^2 + x,\, b)_F \cong [\,0,\, b)_F \cong \mathbb{M}_2(F)$.
(8) $[a,\, b)_F \cong [a,\, bd)_F$ *whenever* $d = x^2 + xy + ay^2 \neq 0$.

Here, (6) follows by considering the generating set $\{i + j,\, j\}$, and (7), (8) follow similarly by considering the generating sets $\{i + x,\, j\}$ and $\{i,\, xj + yk\}$ (where $k = ij$).

We shall also need the following basic observation on quadratic subfields in a quaternion algebra $B$ (see, e.g. [4: p. 104]):

[1] Of course, (4), (5) are also equivalent to (1)–(3) in case $\text{char}(F) \neq 2$, since all quadratic extensions of $F$ are separable in that case. But this would hardly qualify as a "refinement".

(9) *A separable field extension* $F[t]/(t^2+t+a)$ *embeds in* $B$ *iff* $B \cong [a,*)_F$. *An inseparable field extension* $F[t]/(t^2-b)$ *embeds in* $B$ *iff* $B \cong [**,b)_F$.

This observation leads us naturally to the notion of *linkage*. We say that two quaternion algebras $B$, $C$ are *left-linked* if $B \cong [a,x)_F$ and $C \cong [a,y)_F$ for suitable $a \in F$ and $x,y \in F^*$, and *right-linked* if $B \cong [z,b)_F$ and $C \cong [w,b)_F$ for suitable $b \in F^*$ and $z,w \in F$. From (9), we see that, if $B$, $C$ are division algebras, "left-linked" means that they have a common *separable* quadratic subfield, and "right-linked" means that they have a common *inseparable* quadratic subfield.

We shall now prove Draxl's Theorem. (4) $\Leftrightarrow$ (5) being a standard fact on splitting fields, our task at hand is only to prove (3) $\Rightarrow$ (5). In view of the above interpretations of linkage, this implication will follow as soon as we prove the following

**Proposition.** *If two quaternion algebras* $B$, $C$ *are right-linked, then they are left-linked.*

**Proof.** Write $B \cong [z,b)_F$ and $C \cong [w,b)_F$, where $z,w \in F$ and $d \in F^*$. Let $x \in F$ be the unique element solving the linear equation $z + b(x+z) = w$. We may assume that $x^2+x+z \neq 0$ (for otherwise $B$ splits by (7), and hence $B \cong [w,1)_F$ is left-linked to $C$). By (8) and (6),

$$B \cong \left[\,z,\, b(x^2+x+z)\right)_F \cong \left[\,z+b(x^2+x+z),\, *\,\right)_F \cong \left[\,w+bx^2,\, *\,\right)_F. \tag{10}$$

If $x=0$, this shows that $B$ is left-linked to $C$. If $x \neq 0$, then by (8) and (6) again, $C \cong [w,d)_F \cong [w,bx^2)_F \cong [w+bx^2, dx^2)_F$, which is left-linked to $B$ by (10). $\square$

Note that the proof of the Proposition is actually algorithmic: it gives an *explicit* construction of a left linkage from any given right linkage. We finish by showing, however, that the converse of the Lemma is not true; that is, left linkage is in general *weaker* than right linkage. In the language of algebras, this means that it is possible for two quaternion division $F$-algebras to have a common separable quadratic extension of $F$, but no common inseparable quadratic extension.

**Example.** Over the rational function field $F = \mathbb{F}_2(x,y)$, consider the quaternion algebras $B = [1,x)_F$ and $C = [1,y)_F$. These are left-linked, both containing the separable quadratic extension $E/F$ where $E = \mathbb{F}_4(x,y)$. We claim that $B$, $C$ *are not right-linked* (so they do not contain a common inseparable quadratic field extension). To see this, assume instead that $B \cong [*,b)_F$ and $C \cong [**,b)_F$, for some $b \in F^*$. Then, both $B$ and $C$ have a *nonscalar* element with square $b$. A short calculation using the given presentations of $B$ and $C$ leads to the following equations:

$$b = h_1^2 + x(f_1^2+f_1g_1+g_1^2) = h_2^2 + y(f_2^2+f_2g_2+g_2^2),$$

where $(f_i,g_i) \neq (0,0) \in F^2$. Setting $h = h_1+h_2$, we get

$$h^2 = x(f_1^2+f_1g_1+g_1^2) + y(f_2^2+f_2g_2+g_2^2). \tag{11}$$

After clearing denominators, we may assume that $f_i,\, g_i \in \mathbb{F}_2[x,y]$, and with

$$\max\{\deg(f_1),\, \deg(g_1),\, \deg(f_2),\, \deg(g_2)\}$$

chosen as small as possible. Setting $y=0$ in (11), we have

$$h(x,0)^2 = x\,[f_1(x,0)^2 + f_1(x,0)\,g_1(x,0) + g_1(x,0)^2].$$

Since $f_i(x,0)$ and $g_i(x,0)$ are monic (if nonzero), the RHS has odd degree, while the LHS has even degree. Thus, we must have $f_1(x,0) = g_1(x,0) = 0$, so we can write $f_1 = yf_3$ and $g_1 = yg_3$. Similarly, $f_2 = xf_4$ and $g_2 = xg_4$, and hence $h = xyh_3$. Cancelling $xy$ from (11) gives

$$xy\,h_3^2 = y(f_3^2 + f_3g_3 + g_3^2) + x(f_4^2 + f_4g_4 + g_4^2). \tag{12}$$

Repeating the argument gives $f_3 = xf_5$, $g_3 = xg_5$, $f_4 = yf_6$, $g_4 = yg_6$, and now (12) gives

$$h_3^2 = x(f_5^2 + f_5g_5 + g_5^2) + y(f_6^2 + f_6g_6 + g_6^2),$$

which contradicts the minimal choice of $\{f_1, g_1, f_2, g_2\}$ in (11).

Note that $B$, $C$ here are necessarily division algebras, for, if say $B$ was not a division algebra, then $B \cong \mathbb{M}_2(F) \cong [0,\, y)_F$ would have been right-linked to $C = [1,\, y)_F$.

# References


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