Corner Ring Theory: A Generalization of Peirce Decompositions, I

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Abstract

In this paper, we introduce a general theory of corner rings in noncommutative rings that generalizes the classical notion of Peirce decompositions with respect to idempotents. Two basic types of corners are the *Peirce corners* eRe ($e^2 = e$) and the *unital corners* (corners containing the identity of R). A general corner is both a unital corner of a Peirce corner, and a Peirce corner of a unital corner. The simple axioms for corners engender good functorial properties, and make possible a broader study of subrings with only *some* of the features of Peirce corners. In this setting, useful notions such as *rigid corners*, *split corners*, and *semisplit corners* also come to light. This paper develops the foundations of such a corner ring theory, with a view toward a unified treatment of various descent-type problems in ring theory in its sequel.

§1. Introduction

In the fourth volume of the American Journal of Mathematics, Benjamin Peirce published a long article in 1881 with the title "Linear Associative Algebra" [Pe]. Much of Peirce's detailed study of low dimensional associative algebras in this paper is no longer read by current researchers. However, Peirce's realization of the role of nilpotent elements and idempotent elements in the study of an algebra had a lasting impact on (what is later known as) ring theory. On p. 104 of [Pe], Peirce considered an "expression" in an algebra such that, "when raised to a square or higher power, it gives itself as the result"; such an expression, he wrote, "may be called *idempotent*."

Peirce pointed out that a (nonzero) idempotent "can be assumed as one of the independent units" (or basis elements) of the algebra. On p. 109 of [Pe], he wrote:

> "*The remaining units can be selected as to be separable into four distinct groups. With reference to the basis, the units of the first group are idemfactors; those of the second group are idemfaciend and nilfacient; those of the third group are idemfacient and nilfaciend; and those of the fourth group are nilfactors.*"

This quotation from [Pe] seems to be the origin of the *Peirce decomposition* of an algebra R with respect to an idempotent element $e \in R$. Replacing the arcane terminology with modern notations and proceeding in a basis-free manner, we may identify Peirce's four distinct groups as eRe, eRf, fRe, and fRf, where f denotes the "complementary idempotent" $1-e$. The sum of these four additive groups is a direct sum, equal to the whole ring (or algebra) R. The first group eRe is a ring in its own right, with identity element e: this is the *Peirce corner* of R associated to the idempotent e.

Through the last century, the study pf Peirce corners has played a major role in noncommutative ring theory. The use of rings of the type eRe has proved to be important in the consideration of many ring-theoretic issues, such as the decompositions and extensions of rings, continuous geometry, Boolean algebras, projective modules, Morita equivalences and dualities, rings of operators, and path algebras of quivers, etc. However, Peirce corners have a nontrivial presence only in rings with idempotents, so for several important types of rings (e.g. domains or local rings), the theory of Peirce corners cannot be expected to be of any direct impact.

In this paper, we use Peirce corner rings as a model for building a general theory of corner rings in arbitrary rings. The notion of a Peirce corner is generalized as follows. A subring $S \subseteq R$ is called a (general) corner of R if $R = S \oplus C$ for a subgroup $C \subseteq R$ (called a *complement* of S) that is closed with respect to left and right multiplications by elements of S. Peirce corners $R_e := eRe$ are a special case, since C may be taken to be (the "Peirce complement") $C_e := eRf \oplus fRe \oplus fRf$.

The advantage of the corner ring definition above lies in its simplicity and flexibility. Unlike the case of Peirce corners, a general corner $S \subseteq R$ may contain the identity of R without being the whole ring: such S is called a *unital corner* of R. Examples of unital corners are also ubiquitous in ring theory; for instance, they show up as ring retracts, as the 0-th components of monoid graded rings or general crossed products, and (in many significant cases) as rings of invariants with respect to group actions on rings. Peirce corners and unital corners play a special role in our general corner ring theory, since any corner is a unital corner of a Peirce corner, and also a Peirce corner of a unital corner. On the other hand, the simple definition for corner rings in general provides a common axiomatic ground for understanding Peirce corners and unital corners simultaneously.

A nice feature of general corner rings is their very tractable functorial behavior. This is expounded in §2, where we prove (among other things) the transitivity and descent properties of corners ((2.3) and (2.4)). We also characterize Peirce corners and unital corners, respectively, by using properties of their complements ((2.10) and (2.14)). This work, in part, brings forth a main theme of the present paper, namely, that the properties of the complements are as important as those of the corner rings themselves. For another simple illustration, if $R = S \oplus C$ as in the definition of corner rings above, then $C \cdot C = 0$ amounts to R being a "trivial extension" of S, while $C \cdot C \subseteq S$ amounts to R being a $\mathbb{Z}_2$-graded ring, with 0-component S and 1-component C. These and various other examples of corner rings are given in §3, where we draw freely from the many constructions in commutative and noncommutative ring theory.

The general axiomatic formulation of corner ring theory led to some interesting concepts that seemed to have escaped earlier notice. For instance, in studying a general corner ring S, it is natural to ask when S has a *unique* complement, or when S has a complement that is an *ideal* in the ambient ring R. These conditions define the notions of "rigid corners" and "split corners" respectively. For instance, rigid corners occur naturally in the study of crossed products (see (3.6)). In the case of Peirce corners $R_e = eRe$, (2.8) shows that we have automatic rigidity (that is, $C_e = eRf \oplus fRe \oplus fRf$ is the *only* complement for R_e). However, R_e need not split in general; we show (in (4.5) and (4.9)) that it does iff $e(RfR)e = 0$ (where $f = 1 - e$), iff any composition of R-homomorphisms $eR \to fR \to eR$ is zero. In this case, we say that the idempotent e is *split*: this seems to be a useful new notion on idempotents that is worthy of further study. For instance, 1-sided semicentral idempotents (used extensively in studying the triangular representation of rings) are always split, but split idempotents need not be 1-sided semicentral (see (4.10), (4.15)). Incidentally, Peirce corners arising from 1-sided semicentral idempotents are precisely corner rings that are 1-sided ideals, and Peirce corners arising from central idempotents are precisely corner rings that are 2-sided ideals ((2.11) and (2.12)). All of these results serve to show how nicely the classical Peirce corners fit into the general theory of corner rings.

The last section of the paper (§5) is devoted to the aforementioned theme that any corner can be represented as a unital corner of a Peirce corner, and also as a Peirce corner of a unital corner. Along with this work, we prove several results giving a one-one correspondence between the complements of certain pairs of corner rings; see (5.1) and (5.10). The former shows, for instance, that a corner S is rigid in R iff it is rigid in some (or in all) Peirce corner(s) of R containing S.

The conference lecture given in Lisbon (on which this paper is based) also reported on some of the applications of the corner ring theory. This part of our work will appear later in a sequel to this paper, [La$_4$], in which we shall study the multiplicative structure of corner rings and various descent problems of ring-theoretic properties. Some further applications of the viewpoint of corner rings are presented in [LD].

Throughout this note, R denotes a ring with an identity element $1 = 1_R$, and by the word "subring", we shall always mean a subgroup $S \subseteq R$ that is closed under multiplication (hence a ring in its own right), but with an identity element *possibly different from* 1_R. If 1_R happens to be in S (so it is also the identity of S), we say that S is a *unital subring* of R. Other general ring-theoretic notations and conventions in this paper follow closely those used in [La$_1$] and [La$_3$].

§2. Different Types of Corner Rings

We introduce the following general definition of a "corner" in a ring R.

(2.0) Definition. A ring $S \subseteq R$ (with the same multiplication as R, but not assumed to have an identity initially) is said to be a *corner ring* (or simply a corner) of R if there exists an additive subgroup $C \subseteq R$ such that

$$(2.0)' \qquad R = S \oplus C, \quad S \cdot C \subseteq C, \quad \text{and} \quad C \cdot S \subseteq C.$$

In this case, we write $S \prec R$, and we call any subgroup C satisfying $(2.0)'$ a *complement* of the corner ring S in R.

Of course, in general, such a complement C is far from being unique. For instance, if R contains $\mathbb{Z}$ as a unital subring, then any additive subgroup $C \subseteq R$ such that $\mathbb{Z} \oplus C = R$ is a complement of $\mathbb{Z}$ in R in the sense of (2.0). If a corner S of a ring R happens to have a unique complement, we shall call S a *rigid corner of R, and write $S \prec_r R$.*

Proposition 2.1. *A ring $S \subseteq R$ is a corner of R iff the inclusion map $S \hookrightarrow R$ has a splitting $\tau : R \to S$ that is both left and right S-linear.*[1]

Proof. If $\tau : R \to S$ exists, one checks easily that $C := \ker(\tau)$ satisfies $(2.0)'$, so $S \prec R$. Conversely, if $S \prec R$, with a complement C, the inclusion map $S \hookrightarrow R$ splits by the map $\tau : R \to S$ given by $\tau(s + c) = s$ for $s \in S$ and $c \in C$. For $s_0 \in S$, we have

$$\tau(s_0(s + c)) = \tau(s_0 s + s_0 c) = s_0 s = s_0 \tau(s + c)$$

since $s_0 s \in S$ and $s_0 c \in C$. Thus, τ is left S-linear, and a similar check shows that τ is also right S-linear. $\square$

The following easy proposition shows that a corner ring of any ring R must have an identity (although this may not be the identity of R).

(2.2) Proposition. *Let $S \prec R$, with a complement C, and let $1 = e + f$, where $e \in S$ and $f \in C$. Then e is an identity of the ring S. In particular, the decomposition $1 = e + f$ is independent of the choice of the complement C, and e, f are complementary idempotents in R.*

[1] We could have called τ a splitting in the category of (S, S)-bimodules if, by the word "bimodule", we mean a bimodule that is not necessarily unital on either side.

Proof. For any $s \in S$, $s = s \cdot 1 = se + sf$. Since s, $se \in S$ and $sf \in C$, we have $s = se$. Similarly, $s = es$, so e is an identity for S. Since the identity element of S is unique, the remaining statements in the Proposition follow immediately. $\square$

The next two propositions serve to show the robustness of our chosen definition of corners in rings.

(2.3) Proposition. (Descent) *Let $S \prec R$, with a complement C.*

(1) *If S' is any subring of R containing S, then $S \prec S'$ (with complement $C \cap S'$).*

(2) *If R' is any subring of R containing C, then $S \cap R' \prec R'$ (with complement C).*

Proof. (1) Let $C_0 = S' \cap C$. Then $S' = S \oplus C_0$, and

$$S\,C_0 \subseteq S' \cap S\,C \subseteq S' \cap C = C_0\,.$$

By symmetry, we have also $C_0\,S \subseteq C_0$. This shows that $S \prec S'$, with a complement C_0. (2) is proved similarly. $\square$

(2.4) Proposition. (Transitivity) *Suppose $S \prec S'$ and $S' \prec R$. Then $S \prec R$. If $S \prec_r R$, then $S \prec_r S'$.*

Proof. Let C_0 be a complement of S in S' and C' be a complement of S' in R. Then, for $C := C_0 + C'$, we have $S \oplus C = R$. Moreover,

$$S\,C \subseteq S\,C_0 + S\,C' \subseteq C_0 + S'\,C' \subseteq C_0 + C' = C,$$

and similarly $C\,S \subseteq C$. Thus, $S \prec R$, with a complement C. Now assume $S \prec_r R$. If C_1 is another complement of S in S', then $C_1 \oplus C'$ is also a complement of S in R. Therefore, $C_0 \oplus C' = C_1 \oplus C'$. Contracting these to S', we see that $C_0 = C_1$, so $S \prec_r S'$. $\square$

Remark. *If $S \prec R$, a subring $S' \subseteq R$ containing S need not be a corner of R.* For instance, let $S = \mathbb{Z}$, and $R = \mathbb{Z}[x]$ with the relation $x^2 = 0$. Certainly, $S \prec R$, but the subring $S' = \mathbb{Z} \oplus 2\,\mathbb{Z}\,x$ is *not* a corner of R. (If S' has a complement C' in R, take a nonzero element $f = a + bx \in C'$. Then $2f = 2a + 2bx \in S' \cap C' = 0$ implies that $f = 0$, a contradiction.) For some cases in which we can infer that $S' \supseteq S$ is a corner in R, see (5.9).

An easy consequence of (2.3) and (2.4) is the following.

(2.5) Corollary. *If $S \subseteq S'$ are both corners of a ring R, then every complement C' for S' in R can be enlarged to one for S.*

Proof. By (2.3)(1), we know that $S \prec S'$, so we can fix some complement C_0 of S in S'. By the proof of (2.4), $C_0 \oplus C'$ is a complement of S: this is the complement we seek, since it contains C'. $\square$

In view of (2.5), it is natural to ask the following

(2.6) Question. *If $S \subseteq S'$ are both corners of a ring R, does every complement C for S in R contain some complement for S'?*

In general, the answer is "no". We construct a counterexample as follows. For any field k, let R be the commutative local ring $k\,[x]$ with the relation $x^4 = 0$, and let $S = k$, $S' = k\,[x^2]$. Then R has k-basis $\{e_i\}$, where

$$e_1 = 1, \quad e_2 = x^2, \quad e_3 = 1 + x, \quad \text{and} \quad e_4 = 1 + x^3.$$

Thus, the span C of e_2, e_3, e_4 is a complement to S in R. Assume, for the moment, that C contains a complement C' to S'. Since $R = S' \oplus xS'$, we have $C' \cong R/S' \cong S'$ as S'-modules, so there exists an element $y \in C'$ such that C' has a k-basis $\{y,\, x^2y\}$. Write $y = ae_2 + be_3 + ce_4$, where $a, b, c \in k$. Then

$$\begin{aligned} x^2y &= x^2\,[\,ax^2 + b\,(1+x) + c\,(1+x^3)\,] \\ &= (b+c)\,x^2 + bx^3 \\ &= -be_1 + (b+c)e_2 + be_4 \end{aligned}$$

implies that $b = 0$ since $x^2y \in C' \subseteq C$. But then $x^2y = ce_2 \in S'$, which is a contradiction. This shows that C does not contain any complement of S' in R.

In spite of examples such as the above, it turns out that Question (2.6) has an affirmative answer if one of the corners S, S' in question is a Peirce corner. Before we come to the proof of this (in (2.9)), let us first prove a key lemma on complements of general corner rings.

(2.7) Lemma. *Let $S \prec R$, with identity e and complement C, and let $r \in R$. Then $r \in C$ iff $ere \in C$. In particular, $ere = 0 \Longrightarrow r \in C$.*

Proof. The "only if" part follows from (2.0)$'$. For the "if" part, assume that $ere \in C$, and write $r = s + c$, where $s \in S$ and $c \in C$. Then

$$s = ese = e(r - c)e = ere - ece \in C$$

implies that $s = 0$. Thus, $r = c \in C$. $\square$

We now come to the following basic result on Peirce corners.

(2.8) Theorem and Definition (Peirce Corners). *Let e, f be complementary idempotents in a ring R. Then:*

(1) $R_e := eRe \prec R$; *it is the largest subring (resp. corner) of* R *having* e *as identity element.*

(2) $R_e \prec_r R$ *(that is,* R_e *is rigid in* R*), with a unique complement*

$$(2.8)' \qquad C_e := fRe \oplus eRf \oplus fRf = \{r \in R: \; ere = 0\}\,.$$

We shall call R_e *the Peirce corner of* R *(arising from the idempotent* e*), and call* C_e *its Peirce complement. The notations* R_e *and* C_e *will be fixed in the sequel of this paper, and we shall use the notation* $S \prec_P R$ *to refer to the fact that* S *is a Peirce corner of* R*.*

(3) $R_e \cap R' \prec_P R'$ *for any subring* $R' \subseteq R$ *containing* e*.*

(4) *(Transitivity of Peirce Corners) For any subring* $S \subseteq R_e$*,* $S \prec_P R_e$ *iff* $S \prec_P R$*.*

Proof. (1) By Peirce's theory[2], the sum in $(2.8)'$ is direct, and we have $R = R_e \oplus C_e$. An easy calculation shows that

$$(2.8)'' \qquad R_e\, C_e = eRf \subseteq C_e\,, \quad \text{and} \quad C_e\, R_e = fRe \subseteq C_e\,,$$

so $R_e \prec R$, with a complement C_e. If S is any subring of R having e as its identity, then for any $s \in S$, we have $s = ese \in R_e$, so $S \subseteq R_e$.

(2) Consider *any* complement C to R_e. Let $K = \{r \in R: \; ere = 0\}$. Clearly, $C_e \subseteq K$, and by (2.7), $K \subseteq C$. Since C_e and C are *both* complements of R_e, the inclusions $C_e \subseteq K \subseteq C$ must all be equalities! In particular, $R_e \prec_r R$.

(3) Since $e \in R'$, $eR'e \subseteq R' \cap eRe$. On the other hand, for any $r' \in R'$ of the form ere (for some $r \in R$), we have $r' = er'e \in eR'e$. Therefore, $R_e \cap R' = R'_e \prec_P R'$.

(4) The "if" part follows from (3). For the "only if" part, assume that $S \prec_P R_e$. Then $S = e'R_e e'$ where e' is an idempotent in R_e. But then $e'e = e' = ee'$, so $e'R_e e' = e'(eRe)e' = e'Re' \prec_P R$. □

(2.9) Corollary. *Let* $S \subseteq S'$ *be both corners of* R*, with complements* C *and* C' *respectively. If one of* S*,* S' *is a Peirce corner of* R*, then* $C' \subseteq C$*. (In particular,* (2.6) *has an affirmative answer if one of* S*,* S' *is a Peirce corner of* R*.)*

Proof. First assume that $S = eRe$, for some idempotent $e \in R$. Since R_e is rigid, we have here $C = C_e$. By (2.5), C' can be enlarged to a complement of $S = R_e$, which must be C_e. Thus, $C' \subseteq C_e = C$. Next, assume instead, that $S' = e'Re'$, for some idempotent $e' \in R$. Here, $C' = C_{e'}$. Let e be the identity of S. Since $e \in S \subseteq S'$, we have $ee' = e'e = e$. For any $r \in C' = C_{e'}$, we have $e're' = 0$, and hence $ere = ee're'e = 0$. By (2.7), this implies that $r \in C$. Thus, $C' \subseteq C$, as desired. The last conclusion of (2.9) is now obvious. □

[2]For an exposition, see [La_1: p. 308].

For the corollary above, the following remark is relevant. In the first part of the result, the argument would have worked as long as the corner S is rigid. But for the second part of the result, we do require that the corner S' be Peirce. If S' is only rigid, the desired conclusion $C' \subseteq C$ need not hold. For instance, if $R = \mathbb{R}\{1, i, j, k\}$ is the division ring of the real quaternions, then $S = \mathbb{R} \prec R$ is contained in $S' = \mathbb{R}\{1, i\} \prec R$. Here, $C' := \mathbb{R}\{j, k\}$ is a complement of S', $C = \mathbb{R}\{i,\, j,\, 1+k\}$ is a complement of S, but $C' \not\subseteq C$. (It is easy to see that S' here is indeed rigid in R. For a more general fact, see (3.6) below.)

We can give some easy characterizations of Peirce corners, as follows.

(2.10) Proposition. *For $S \prec R$ with identity e, the following are equivalent:*

(1) *$S \prec_P R$;*
(2) *S has a complement C such that $eCe = 0$;*
(3) *the subring $S \subseteq R$ is "hereditary", in the sense that $sRs' \subseteq S$ for all $s,\, s' \in S$.*

Proof. (1) ⇒ (2) follows by taking C to be a Peirce complement (in case $S \prec_P R$).
(2) ⇒ (1). If C is as in (2), then

$$eRe = e(S+C)e = eSe + eCe = eSe = S \implies S = R_e \prec_P R.$$

(1) ⇒ (3). If $S = R_e$, we have $s = es$ and $s' = s'e$ for any $s,\, s' \in S$. Therefore, $sRs' = esRs'e \subseteq eRe = S$.
(3) ⇒ (1). It suffices to assume only $sRs \subseteq S$ for all $s \in S$. For, if so, then $e \in S$ implies that $eRe \subseteq S$, and we must have equality here since $S = eSe \subseteq eRe$. □

After my lecture at the Lisbon Conference, Professors J. Okniński and F. Perera both pointed out to me the importance of the notion of hereditary subalgebras in algebra and in analysis. This prompted me to include the result (3) above, according to which the hereditary corner rings (in our general sense) are precisely the classical Peirce corner rings.

Next, we shall characterize corners in R that are one-sided ideals. These turn out to be necessarily Peirce corners, but they are Peirce corners of a special kind. To see this, let us first recall some standard definitions in the theory of idempotents. In [Bi], [BH], and [HT], an idempotent $e \in R$ with complementary idempotent $f = 1 - e$ is said to be *left semicentral* if $fRe = 0$, and *right semicentral* if $eRf = 0$. Take, for instance, the former: it is easy to show that

$$(*) \qquad \begin{aligned} fRe = 0 &\iff f(ReR)e = 0 \\ &\iff ere = re \ (\forall\, r \in R) \iff eRe = Re \\ &\iff eR \text{ is an ideal} \iff Rf \text{ is an ideal,} \end{aligned}$$

so each of these conditions is a characterization for e to be a left semicentral idempotent. A similar remark applies to right semicentral idempotents.

(2.11) Proposition. *For a corner $S \prec R$ with identity e and complement C, the following are equivalent:*

(1) *S is a left ideal in R;*
(2) $C \cdot S = 0$;
(3) *e is a left semicentral idempotent in R and $S = R_e$.*

Proof. (1) $\Leftrightarrow$ (2). If S is a left ideal, then $C \cdot S \subseteq S \cap C = 0$. Conversely, if $C \cdot S = 0$, then $R \cdot S = (S + C) \cdot S = S \cdot S = S$, so S is a left ideal.

(3) $\Rightarrow$ (1). If $S = R_e$ with e left semicentral, then by $(*)$ above, $S = eRe = Re$, so it is a left ideal.

(1) $\Rightarrow$ (3). If S is a left ideal, then $e \in S$ yields $Re \subseteq S \subseteq eSe \subseteq eRe$. Therefore, equality holds throughout, so $S = eRe$, and $Re = eRe$ implies that e is left semicentral by $(*)$. ☐

If the idempotent e is *both* left and right semicentral, then $re = ere = er$ for all $r \in R$, so e is in fact central. Then $R_e = eR$ and $C_e = R_f = fR$ are both ideals of R, so R is a ring direct product $R_e \times R_f$. In this case, we call $R_e = eR$ a *direct Peirce corner* of R. We can thus conclude:

(2.12) Corollary. *A corner $S \prec R$ is a direct Peirce corner iff S is an ideal of R.*

We move on now to consider another important type of corner rings.

(2.13) Definition (Unital Corners). Let S be a corner of R, with identity e. We say that S is a *unital corner* (and write $S \prec_u R$) if $e = 1$ (that is, if S is a unital subring of R).

It is easy to see that $S \prec_P R$ and $S \prec_u R$ iff $S = R$. In parallel to (2.10), the following is a characterization of unital corners.

(2.14) Proposition. *For $S \prec R$ with an identity e, the following are equivalent:*

(1) *$S \prec_u R$;*
(2) *S has a complement C with $eCe = C$;*
(3) *every complement C of S satisfies $eCe = C$.*

Proof. (1) $\Rightarrow$ (3) is clear, since $e = 1$ under (1). (3) $\Rightarrow$ (2) is also clear, since a complement of S always exists. To prove (2) $\Rightarrow$ (1), suppose S has a complement C with $eCe = C$. Then, e acts as the identity map by left and by right multiplication on C, as well as on S. This clearly implies that $e = 1$; that is, $S \prec_u R$. ☐

We shall now introduce two more kinds of corner rings.

(2.15) Definition (Split and Rigid-Split Corners). A corner S in R is called a *split corner* (written $S \prec_s R$) if it has a complement C that is an ideal in R. Note that, in view of $(2.0)'$, this is equivalent to S having a complement C that is closed under multiplication. In this case, we have a unital ring isomorphism $S \cong R/C$, although, as a subring of R, S may still be not unital. If $S \prec_s R$ happens to have a unique ideal complement, we shall call it a *rigid-split corner,* and write $S \prec_{rs} R$.

A word of caution is necessary on this piece of terminology. If a corner S in R is both rigid and split, then clearly $S \prec_{rs} R$. However, if $S \prec_{rs} R$, then S is split, *but it may not be rigid,* as there may be other complements to S besides the guaranteed unique ideal complement. For an example illustrating this situation, see (3.9) below.

For split corners, it is easy to verify the following analogues of (2.3) and (2.4) (for descent and transitivity), and of (2.5).

(2.16) Proposition. (1) *If $S \prec_s R$, then $S \prec_s S'$ for any subring S' of R containing S. And, for any subring R' containing an ideal complement of S, $S \cap R' \prec_s R'$.*
(2) *$S \prec_s S'$ and $S' \prec_s R$ imply $S \prec_s R$. If $S \prec_{rs} R$, then $S \prec_{rs} S'$ for any subring $S' \supseteq S$.*
(3) *If $S \subseteq S'$ are both split corners of R, then any ideal complement of S' can be enlarged into one for S.*

Proof. (1) is obvious as an ideal complement C of S in R contracts to an ideal complement of S in S', and C remains an ideal complement to $S \cap R'$ in R'. To prove (2), take C_0 to be an ideal complement of S in S', and C' to be an ideal complement of S' in R. Then, for the complement $C := C_0 + C'$ of S in R, we have

$$RC = R\,(C_0 + C') \subseteq (S' + C')\,C_0 + C' \subseteq C_0 + C' = C,$$

and similarly, $CR \subseteq C$. Thus, C is an ideal in R, so $S \prec_s R$. The second part of (2) can be proved in the same way as the second part of (2.4): in the argument there, we simply replace complements by ideal complements. The proof for (3) follows by a similar modification of that for (2.5). $\square$

Split unital corners in a ring R are very familiar objects in ring theory; they are called "retracts" of R. On the other hand, split Peirce corners did not seem to have received much attention; we shall study them in more detail in §4. Here, let us give an example of a split corner that is neither unital nor Peirce.

(2.17) Example. Take a split unital corner S_0 in some ring A, with an ideal complement $C_0 \neq 0$, and let $R = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$. Then, for $S := \begin{pmatrix} S_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $C := \begin{pmatrix} C_0 & A \\ 0 & A \end{pmatrix}$, we have $R = S \oplus C$, and a direct calculation shows that C is a complement of S in R, with $C \cdot C \subseteq C$. Thus, $S \prec_s R$, with the identity element $e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

This is not the identity of R, so S is not a unital corner. It is also *not* a Peirce corner, since $R_e = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ properly contains $S = \begin{pmatrix} S_0 & 0 \\ 0 & 0 \end{pmatrix}$.

In ring theory, there is a very useful construction of "trivial extensions", whereby, for any ring S and a unital (S,S)-bimodule C, a ring $R := S \oplus C$ is produced in which S is a unital subring, $C \cdot C = 0$, and the left/right multiplications of S on C are given by the (S,S)-bimodule structure. Such a ring R is called a *trivial extension* of S by $C = {}_SC_S$; see [La$_3$: p. 37]. Clearly, S is a retract of R, and C is a complement of S with a trivial multiplication. Note that R may also be viewed as the unital subring of the triangular ring $\begin{pmatrix} S & C \\ 0 & S \end{pmatrix}$ consisting of matrices of the form $\begin{pmatrix} s & c \\ 0 & s \end{pmatrix}$, with $s \in S$ and $c \in C$.

To relate the construction of trivial extensions to corner rings, we make the following:

(2.18) Definition (Trivial Corners). A corner S of a ring R is said to be a *trivial corner* if it has a complement C with $C \cdot C = 0$.

(2.19) Proposition. *A trivial corner S of R is a retract of R, and R is a trivial extension of S.*

Proof. Say C is a complement of S with $C \cdot C = 0$. Then C is an ideal of R, so S is a split corner. Let $1 = e + f$, where $e \in S$ and $f \in C$. Then $f = f \cdot f \in C \cdot C = 0$ leads to $e = 1$. Thus, S is a unital corner, and hence a retract of R. Here, under the ring structure on R, C is a unital (S,S)-bimodule, and, for s, $s' \in S$ and c, $c' \in C$:

$$(s + c)(s' + c') = ss' + sc' + cs',$$

so R is precisely the trivial extension of S by ${}_SC_S$. $\square$

In conclusion, let us also point out that standard constructions in ring theory can be used to give various examples of new corners from old ones. We note for instance the following three types of constructions, starting from any $S \prec R$ with a complement C.

(2.20) For any integer $n \geq 1$, it is routine to check that $\mathbb{M}_n(C)$ is a complement to $\mathbb{M}_n(S)$ in $A := \mathbb{M}_n(R)$. Thus, $\mathbb{M}_n(S) \prec A$. If $S \prec_P R$, say $S = R_e$ in R, then $\mathbb{M}_n(S) \prec_P A$. In fact, an easy calculation shows that $\mathbb{M}_n(S) = A_e$, where e is viewed as usual as the (idempotent) matrix $eI_n \in A$. (The same applies to rings of upper triangular matrices.)

(2.21) $S\,[(x_i)_{i \in I}] \prec R\,[(x_i)_{i \in I}]$; a complement is given by $C\,[(x_i)_{i \in I}]$. (The same applies to power series constructions.)

(2.22) For any (multiplicative) monoid G, the monoid ring $S[G] \prec R[G]$; a complement is given by $C[G]$.

§3. Examples of Unital Corners (and Their Complements)

Peirce corners are easy to find since they are as ubiquitous as idempotents in rings. However, a Peirce corner cannot be unital unless it is the whole ring. In (3.1) below, we shall collect some examples of unital corners (and their complements). Note that, *if R is a ring with only trivial idempotents $\{0,1\}$, then any nonzero corner ring S is necessarily unital.* (The identity e of S is a nonzero idempotent, and so $e = 1$.) One major difference between Peirce corners and unital corners is the following: by (2.8)(2), Peirce corners are rigid, but, as we'll see from the examples below, unital corners need not be rigid, and split unital corners need not be rigid either.

(3.1) Examples.

(A) Let S be a *central unital* subring of R. If S is a self-injective ring or ${}_S R$ is a semisimple S-module, then $S \prec_u R$. In fact, in either case, we have $R = S \oplus C$ for a suitable submodule C of ${}_S R$. Since S is central, we have $CS = SC \subseteq C$, so $S \prec_u R$.

(B) Let H be a subgroup of a group G. Then, for any ring k, $kH \prec_u kG$. In fact, a complement for kH can be taken to be $C = \bigoplus_{g \notin H} kg$.

(C) Let S be a unital subring of a nonzero ring R. If R is free as a left S-module with a basis G containing 1 such that $gS \subseteq Sg$ for every $g \in G$, then $S \prec_u R$, with a complement $\bigoplus_{g \neq 1} Sg$. For instance, for any base ring k, take R to be the polynomial ring $k\,[x]$. Then, for any $n \geq 1$, $S = k\,[x^n] \prec_u R$ since R is a free (left, right) S-module on the basis $\{1, x, \ldots, x^{n-1}\}$.

(D) For $R = \mathbb{M}_n(k)$ where k is any ring, let S be the subring of diagonal matrices in R. Then $S \prec_u R$, with a complement C given by the group of all matrices with a zero diagonal. (It is easy to check that $SC \subseteq C$ and $CS \subseteq C$.) It can be seen, actually, that this is a special case of (C): for instance, for $n = 3$, we can take G in (C) to be the 3-element set consisting of

$$I_3 = e_{11} + e_{22} + e_{33}, \quad e_{12} + e_{23} + e_{31}, \quad \text{and} \quad e_{13} + e_{21} + e_{32}\,.$$

(E) Another interesting special case of (C) above is a monoid ring $R = S\,[G]$, where G is any multiplicative monoid (and S is some ring). If G has an invertible element $g \neq 1$, then the complement $C := \bigoplus_{g \neq 1} Sg$ (given in (C) above) is *not* an ideal, since $C \cdot g^{-1} \nsubseteq C$ (if $S \neq 0$). Nevertheless, S is a split corner, since we can choose, as another complement, the augmentation ideal $\sum_{g \neq 1} S(g-1)$. This shows that *not every complement of a split unital corner need to be an ideal.* More generally, if R is a ring graded by a monoid G, say $R = \bigoplus_{g \in G} R_g$ (with $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$), then $R_1 \prec_u R$, with complement $C := \bigoplus_{g \neq 1} R_g$. If $gh = 1 \in G$ can happen only if $g = h = 1$, then C is an ideal, so in fact $R_1 \prec_s R$. On the other hand, if we consider the case $G \cong \mathbb{Z}_2$, *a* $\mathbb{Z}_2$*-graded ring corresponds exactly to a ring* R *with a given unital corner* $S \subseteq R$ *having a complement* C *such that* $C \cdot C \subseteq S$.

(F) Various twisted versions of the monoid ring example in (E) (of the "crossed product" variety) also give examples of unital corners. We shall only mention (here and in (G) below) two of the most basic types of examples. Let S be a ring with an endomorphism $\sigma : S \to S$, and let R be the skew polynomial ring $S[x; \sigma]$, whose elements have the form $\sum_i a_i x^i$ $(a_i \in S)$, and are multiplied by the rule $xa = \sigma(a)x$. Here, we have $x^i S \subseteq \sigma^i(S)\, x^i$ for every $i \geq 0$, so $S \prec_u R$, with a complement

$$C := \left\{ \sum_{i \geq 1} a_i x^i : \ a_i \in S \right\} = \bigoplus_{i=1}^{\infty} S\, x^i.$$

This complement is an ideal, so $S \prec_s R$. This example serves to show the existence of nonrigid split corners. For instance, taking $\sigma = \mathrm{Id}_S$, we can choose a new variable $y = x + a$ (with a in the center of S) and write $R = S[y]$. With respect to this expression, $C' = \bigoplus_{i=1}^{\infty} S y^i$ is another ideal complement of S. In case S has an infinite center, this gives infinitely many ideal complements to S.

(G) Yet another interesting special case of (C) is the following example from the theory of central simple algebras. Let R be a central simple algebra of degree d over a field F containing a subfield K that is Galois over F of dimension d. It is well-known that R can be written as a crossed product algebra $\bigoplus_{g \in G} K u_g$, where $G = \mathrm{Gal}(K/F)$, with $u_1 = 1$, $u_g k = g(k) u_g$, and $u_g u_h \in K^* u_{gh}$ (for $g, h \in G$ and $k \in K$). In particular, we have $u_g K = K u_g$ for all $g \in G$. Thus, $K \prec_u R$, with a complement $C := \bigoplus_{g \in G \setminus \{1\}} K u_g$. More general crossed products (in the sense of [Pa]) can be treated similarly.

(H) If R is a *commutative* ring, there is an interesting criterion for a unital subring $S \subseteq R$ to be a corner; namely,

$$S \prec_u R \iff \mathrm{tr}({}_S R) = S. \tag{3.2}$$

Here, the "trace ideal" $\mathrm{tr}({}_S R)$ is the sum of the images of all S-linear functionals on the S-module R. This criterion is due to G. Azumaya and B. Müller; a proof of it can be found in [La_3: (2.49)]. From this, it can be shown that, if R is a finitely generated projective module over S, or if ${}_S R$ is projective and S is a noetherian ring, then $S \prec_u R$ (see [La_3: (2.50)]). Thus, for instance, if S is a Dedekind domain, then S is a corner in any domain $R \supseteq S$ that is finitely generated as an S-module.

(I) Let G be a finite group acting on a ring R, and let $S = R^G$ be the subring of G-invariant elements of R. *If* $|G|^{-1}$ *exists in* R, *then* $S \prec_u R$. To see this, let $\tau : R \to S$ be (the "averaging map") defined by

$$\tau(r) = |G|^{-1} \cdot \sum_{g \in G} r^g \in S. \tag{3.3}$$

This map clearly splits the inclusion $S \hookrightarrow R$, and for $s \in S$, we have $\tau(sr) = s\tau(r)$ (since $(sr)^g = s^g r^g = s \cdot r^g$), and similarly $\tau(rs) = \tau(r)s$. Thus, $S \prec_u R$ by (2.1).[3] Furthermore, S can be thought of, *in two different ways*, as a corner in the skew group ring $A := R * G$. Here, $R * G$ consists of (finite) formal sums $\sum_{g \in G} r_g\, g$ $(r_g \in R)$, which are multiplied by using the rule $gr = r^{g^{-1}} g$ for $r \in R$ and $g \in G$. Upon identifying $r \in R$ with $r \cdot 1 \in A$, we have $R \prec_u A$ (with complement $\sum_{g \neq 1} Rg$), so the transitivity property (2.4) implies that $S \prec_u A$. Secondly, let e be the idempotent $|G|^{-1} \sum_{g \in G} g$ in A, and let

(3.4) $\qquad \varphi : S \to eAe$ be defined by $\varphi(s) = es = se = ese \quad (\forall\ s \in S)$.

It is easy to check that φ is a ring isomorphism: see [Mo: Lemma 2.1], or [Al]. Thus, $S = R^G$ *is also isomorphic to the Peirce corner* eAe *in* A. These examples of corner rings in R and in $R * G$ set the stage for some useful applications of corner ring theory to the study of rings of the type R^G; such applications will be more fully explored in Part II of this paper ([La$_4$]).

(J) If R is a finite von Neumann algebra and S is the center of R, then the center-valued trace $\Delta : R \to S$ is S-linear and is the identity on S (see [KR: (8.4.3)]). Therefore, by (2.1), S is a (unital) corner of R.

Let us now give some examples of *rigid* unital corners, partly drawing from the list of examples above. Specifically, consider the unital corners arising in the manner of (3.1)(C). If R is a commutative ring, then in the notations there, we can produce other complements for S by changing the given S-basis $G = \{1, g_1, g_2, \dots\}$ to, say, $\{1,\ s_1 + g_1,\ s_2 + g_2,\ \dots\}$ (where $s_i \in S$). Therefore, we do not expect the unital corner S to be rigid in this case. However, if R is noncommutative, our odds are better, as the following three examples show.

(3.5) Example. *The corner* $S \prec_u R = \mathbb{M}_n(k)$ *in* (3.1)(D) *is rigid.* To see this, consider any complement C_0 for S, and use the notations in (3.1)(D). Let $a \in k$, and let i, j be two distinct indices in $\{1, \dots, n\}$. Then $a\, e_{ij} = \mathrm{diag}(b_1, \dots, b_n) + M$ for some $b_1, \dots, b_n \in k$ and $M \in C_0$. Thus, $M = a\, e_{ij} - \sum_\ell b_\ell\, e_{\ell\ell}$. Since e_{ii}, $e_{jj} \in S$, C_0 contains the matrix

$$e_{ii}\, M\, e_{jj} = a\, e_{ii}\, e_{ij}\, a_{jj} - \sum_\ell b_\ell\, e_{ii}\, e_{\ell\ell}\, e_{jj} = a\, e_{ij}\,.$$

This shows that C_0 contains the group C of *all* matrices with a zero diagonal, and hence $C_0 = C$, proving that S is rigid in $R = \mathbb{M}_n(k)$.

[3] Analogues of this also exist for various actions of *infinite* groups G on R. In the case where R is commutative, for instance, it is often possible to replace the averaging map τ in (3.3) by a suitable "Reynolds operator" (an R^G-linear retraction from R to R^G). Cayley's "Ω process" and Weyl's "unitarian trick" are among the best known examples of this in classical invariant theory.

(3.6) Example. *The corner* $K \prec_u R$ *in the crossed product example in* (3.1)(G) *also turns out to be rigid.* To see this, consider any complement C_0 for K, and keep the notations in (3.1)(G). To show that $C_0 = C$, we exploit the same ideas used in the usual proof for the simplicity of the crossed product algebra R. Given any $g \neq 1$ in G, decompose the element u_g into $a + c$, where $a \in K$ and $c \in C_0$. Take any element $k \in K$ such that $g(k) \neq k$. Then C_0 contains the element

$$kc - ck = (k\,u_g - ka) - (u_g k - ak) = ku_g - g(k)u_g = (k - g(k))\,u_g\,.$$

Since $k - g(k) \in K^*$, left multiplication of the above element by $(k - g(k))^{-1}$ yields $u_g \in C_0$. Thus, C_0 contains $C = \bigoplus_{g \in G \setminus \{1\}} K u_g$, and hence $C_0 = C$, proving that K is rigid in R. (Again, the case of more general crossed products can be treated similarly.)

(3.7) Example. Let k be any ring, and $R = k\langle x, y\rangle$ with the relations $yx = y^2 = 0$. Then $R = S \oplus Sy$, where $S = k[x]$. Here, $C := Sy$ is an ideal with square zero, so S is a trivial unital corner in R, with complement C. We claim that S is rigid. Indeed, if D is another complement, let $y = s + d$, where $s \in S$ and $d \in D$. Then $0 = yx = sx + dx$ implies that $sx = 0$, so $s = 0$, and $y = d \in D$. It follows that $C = Sy \subseteq SD \subseteq D$, so $C = D$, proving that $S \prec_r R$.

Of course, there are good examples of *rigid-split* unital corners too. Some of the most natural examples are given by the following result.

(3.8) Proposition. *Let S be a (not necessarily unital) corner of a ring R, with a complement C. If* $C = \mathrm{rad}(R)$ (*the Jacobson radical*), $\mathrm{Nil}_*(R)$ (*the lower nilradical*), *or* $\mathrm{Nil}^*(R)$ (*the upper nilradical*), *then* $S \prec_{rs} R$.

Proof. Suppose C' is another ideal complement of S. We would like to prove that $C \subseteq C'$, for then $C' = C$, and we'll have $S \prec_{rs} R$. First assume that $C = \mathrm{rad}(R)$. Let $\pi : R \to R/C'$ be the projection map modulo C'. Then $R/C' \cong S \cong R/C$ implies that $\mathrm{rad}(R/C') = 0$. Since the surjection π takes $\mathrm{rad}(R)$ into $\mathrm{rad}(R/C')$, it follows that $C = \mathrm{rad}(R) \subseteq \ker(\pi) = C'$, as desired. The two cases $C = \mathrm{Nil}_*(R)$ or $\mathrm{Nil}^*(R)$ can be handled similarly. □

(3.9) Corollary. (1) *If a local ring* $(R, \mathfrak{m})$ *has a subring S that maps isomorphically onto the residue division ring* $R/\mathfrak{m}$, *then* $S \prec_{rs} R$, *with a unique ideal complement* $\mathfrak{m}$. (2) *If a semiprime ring* $S \prec R$ *has a complement that is a nilpotent ideal in* R, *then* $S \prec_{rs} R$. *(In particular, a semiprime ring S is always a rigid-split corner in any trivial extension of S.)*

(3.10) Example. (3.9)(1) above gives a natural source for examples of rigid-split corners that are not "rigid and split". For instance, let $R = \mathbb{Q}[x]$, with the relation $x^2 = 0$. Then R is a local ring with maximal ideal $\mathbb{Q}\,x$. By (3.9)(1), $\mathbb{Q} \prec_{rs} R$, with a unique ideal complement $\mathbb{Q}\cdot x$. But $\mathbb{Q}$ is *not* a rigid corner; in fact, it has infinitely many complements $\mathbb{Q}\cdot(x-a)$ for a ranging over $\mathbb{Q}$.

(3.11) Example. The conclusion in (3.9)(2) is in general not true if S is not semiprime. For instance, if $R = \mathbb{Z}_4[x]$ with the relation $x^2 = 0$, then R is a trivial extension of $S = \mathbb{Z}_4$ by its ideal complement $C = \mathbb{Z}_4\cdot x$. Here, $C' = \mathbb{Z}_4\cdot(\bar{2}+x)$ is easily checked to be another ideal complement to S (also with square zero, since $(\bar{2}+x)^2 = \bar{4} + \bar{4}\,x + x^2 = 0$). Thus, S is not a rigid-split corner in R.

(3.12) Example. We close by mentioning that some examples of rigid-split (unital) corners can also be gotten from group ring constructions in (E) above. For any commutative ring k, *the subring k is a rigid-split (unital) corner of a group ring kG iff there is no nontrivial homomorphism from G to the group of units* $\mathrm{U}(k)$. In this case, the augmentation ideal in kG is the unique ideal complement for k. (The proof of this is left as an easy exercise.) Thus, for instance, if G is a group with no subgroup of index 2, then $\mathbb{Z}$ is a rigid-split unital corner in $\mathbb{Z}G$.

§4. Split Peirce Corners

Once the notion of corners is formulated, we have the associated notion of split corners, and in particular, split Peirce corners. Prior to this, however, split Peirce corners did not seem to have been fully scrutinized. In this section, we shall prove a few basic facts about split Peirce corners, some of which will be generalized later to arbitrary split corners.

(4.1) Theorem. *Given a Peirce corner R_e ($e = e^2$), let $\langle C_e\rangle$ be the ideal of R generated by C_e. Then, for $f = 1-e$, we have the equations*

(4.2) $$\langle C_e\rangle = RfR = e(RfR)e \oplus C_e\,,$$

and a ring isomorphism $R/RfR \cong eRe/e(RfR)e$.

Proof. Since $f \in C_e$, clearly $RfR \subseteq \langle C_e\rangle$. On the other hand, $C_e \subseteq RfR$, so $\langle C_e\rangle \subseteq RfR$. This proves the first equality in (4.2). As for the second equality, the inclusion "$\supseteq$" is clear, and "$\subseteq$" will follow if we can show that $e(RfR)e \oplus C_e$ is an ideal of R. This is a routine check that we can safely leave to the reader. Finally, $e(RfR)e \cap C_e \subseteq eRe \cap C_e = 0$, so the sum on the RHS of (4.2) is direct. The last conclusion of the Proposition follows from the Noether Isomorphism Theorem, as $eRe + RfR = R$, and $eRe \cap RfR = e(RfR)e$. $\square$

We record below a couple of natural consequences of (4.1).

(4.3) Corollary. *Recall that an idempotent $f \in R$ is said to be full if $RfR = R$. This is the case iff $e(RfR)e = eRe$ (where $e = 1-f$).*

Proof. This follows from the last conclusion of (4.1). $\square$

(4.4) Corollary. (1) *$ReR \cap RfR = e(RfR)e \oplus eRf \oplus fRe \oplus f(ReR)f$. In particular, the* RHS *is an ideal in R.*
(2) *We have a ring isomorphism:*

$$R/[e(RfR)e \oplus eRf \oplus fRe \oplus f(ReR)f] \cong (R/ReR) \times (R/RfR).$$

Proof. (1) The inclusion "$\supseteq$" is clear. To prove "$\subseteq$", consider any element $r \in ReR \cap RfR$. Write $r = a + b + c + d$ where $a \in eRe$, $b \in eRf$, $c \in fRe$ and $d \in fRf$. After modifying r by $b + c \in eRf \oplus fRe \subseteq ReR \cap RfR$, we are reduced to handling the case $r = a + d$. Now $a = r - d \in eRe \cap RfR = e(RfR)e$, and $d = r - a \in fRf \cap ReR = f(ReR)f$, so $r \in e(RfR)e \oplus f(ReR)f$, which completes the proof of (1).

(2) $ReR + RfR$ is the unit ideal, since it contains $e + f = 1$. Thus, (2) follows from (1) and the Chinese Remainder Theorem. $\square$

(4.5) Theorem (Split Idempotent Criteria). *For any idempotent $e \in R$, the following conditions are equivalent:*

(1) $R_e \prec_s R$;
(2) $R_e \prec_{rs} R$;
(3) *C_e is an ideal of R;*
(4) *$e(RfR)e = 0$;*
(5) *$exeye = exye$ for all $x, y \in R$;*
(6) *the map $\varphi : R \to R_e$ defined by $\varphi(x) = exe$ (for any $x \in R$) is a (unital) ring homomorphism.*

If any of these conditions holds, we say that e is a split idempotent of R.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). By (2.8)(2), R_e has a unique complement C_e. Thus, R_e has an ideal complement iff C_e is an ideal of R, in which case R_e is automatically rigid-split.

(3) $\Leftrightarrow$ (4). Note that (2) holds iff $\langle C_e \rangle = C_e$. By (4.2), this holds iff $e(RfR)e = 0$.

(4) $\Leftrightarrow$ (5). This is clear since (4) amounts to $ex(1 - e)ye = 0$ for all $x, y \in R$.

(5) $\Leftrightarrow$ (6) is also clear, since φ is always additive (and unital), and (5) amounts to the fact that φ is multiplicative. $\square$

(4.6) Corollary. *Let e be a split idempotent, and let $f = 1 - e$. Then* (1) *f is not full unless $e = 0$, and* (2) *e is not full unless $e = 1$.*

Proof. (1) If $RfR = R$, then (4.5)(4) implies that $eRe = 0$, and so $e = 0$. For (2), assume that $ReR = R$. Then

$$fRe \subseteq (ReR)fRe \subseteq R{\cdot}e(RfR)e = 0$$

by (4.5)(4), and similarly, $eRf = 0$. Thus, e is a *central* idempotent. But then $R = ReR$ implies that $e = 1$. $\square$

As a quick example, in a matrix ring $R = \mathbb{M}_n(k)$ with $n \geq 2$ and $k \neq 0$, the matrix unit $e := e_{11}$ is a *nonsplit* idempotent, and its complementary idempotent $f = e_{22} + \cdots + e_{nn}$ is full. Note that, in some cases, we may have a partial converse to (4.6)(1); for instance, for simple Peirce corners, we have the following.

(4.7) Corollary. *If eRe is a simple ring, then $e \in R$ splits iff $f = 1 - e$ is not a full idempotent.*

Proof. The "only if" part follows from (4.6) since $e \neq 0$. Conversely, if f is not full, then by (4.3), $e(RfR)e \neq eRe$. Since $e(RfR)e$ is an ideal of the simple ring eRe, we must have $e(RfR)e = 0$, and so e splits by (4.5). $\square$

It turns out that the condition $e(RfR)e = 0$ in (4.5)(4) has another nice interpretation in terms of R-module homomorphisms. To formulate the ideas more broadly, we take the viewpoint that any ring R is the endomorphism ring of some (say right) module over some other ring (e.g. the right module R_R).

(4.8) Proposition. *Let $R = \mathrm{End}(M_A)$, where M_A is a right module over some ring A. Let $M = P \oplus Q$ be a direct sum decomposition of M_A, and let $e, f \in R$ be, respectively, the projections of M onto P and Q with respect to this decomposition. Then $e(RfR)e = 0$ iff the composition*

$$P \xrightarrow{\alpha} Q \xrightarrow{\beta} P$$

is zero for any A-homomorphisms $\alpha : P \to Q$ and $\beta : Q \to P$.

Proof. Define a map $\varphi : fRe \to \mathrm{Hom}_A(P, Q)$ by $\varphi(g) = g\,|P$. This is an additive group isomorphism, since it has an inverse φ' given by $\varphi'(h) = h'$ where $h' \in R$ denotes the extension of $h : P \to Q \subseteq M$ with $h'(Q) = 0$. (It is easy to see that $h' \in fRe$.)[4] Similarly, we have an additive group isomorphism $\psi : eRf \to \mathrm{Hom}_A(Q, P)$ defined by restriction to Q. If we think of the isomorphisms φ and ψ as "identifications", the condition $0 = e(RfR)e = (eRf)(fRe)$ translates into the statement that any composition of A-homomorphisms $P \to Q \to P$ is zero. $\square$

(4.9) Corollary. *An idempotent $e \in R$ with complementary idempotent f is split iff any composition of R-homomorphisms $eR \to fR \to eR$ is zero.*

Proof. This follows by applying the Proposition to the case $R = \mathrm{End}\,(R_R)$ and taking $B = eR$, $C = fR$. $\square$

[4]This implies, incidentally, that the idempotent e is left semicentral in R iff $\mathrm{Hom}_A(P, Q) = 0$.

(4.10) Corollary. *If $e = e^2$ is left semicentral, then e and $f = 1 - e$ are split idempotents. Moreover, $eRf = ReR \cap RfR$ is an ideal, and we have a ring isomorphism $R/eRf \cong eRe \times fRf$. (Thus, as long as $eRf \neq 0$, e, f are non-central and non-full idempotents.)*

Proof. From $fRe = 0$, we have of course $e(RfR)e = f(ReR)f = 0$. Thus, e and f are both split according to (4.5).[5] Furthermore, (4.4)(1) simplifies to $ReR \cap RfR = eRf$, so eRf is an ideal of R. The isomorphism $R/eRf \cong eRe \times fRf$ follows from the Peirce decomposition (and is, in fact, a special case of the isomorphism in (4.4)(2)). $\square$

Of course, the second part of this Corollary also follows easily from the usual representation of R as a formal triangular ring $\begin{pmatrix} eRe & eRf \\ 0 & fRf \end{pmatrix}$, where eRf is viewed as an $(eRe,\, fRf)$-bimodule in the obvious way (by multiplication in R). In general, if S, T are rings and $M = {}_SM_T$ is an (S,T)-bimodule, then the triangular ring $R := \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ has the property $fRe = 0$ for the complementary idempotents $e = \begin{pmatrix} 1_S & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 0 & 1_T \end{pmatrix}$ in R. Here, $eRf = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \neq 0$ if $M \neq 0$.

The following examples show that, for two complementary idempotents e, $f \in R$, the splittings of e and f are, in general, *independent* conditions. The same examples also show that it is possible for e and/or f to be split without being left or right semicentral.

(4.11) Example. Take a ring A with a pair of complementary idempotents ε, ε' such that $\varepsilon' A \varepsilon = 0 \neq \varepsilon A \varepsilon'$ (that is, ε is left semicentral but not right semicentral), and let $R = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$. Consider in R the complementary idempotents $e = \begin{pmatrix} \varepsilon' & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$. These are *not* one-sided semicentral in R, since

$$fRe = \begin{pmatrix} \varepsilon A \varepsilon' & 0 \\ 0 & 0 \end{pmatrix} \neq 0, \quad \text{and} \quad eRf = \begin{pmatrix} \varepsilon' A \varepsilon & \varepsilon' A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon' A \\ 0 & 0 \end{pmatrix} \neq 0.$$

Taking the products of these, we see that $e(RfR)e = 0$, but $f(ReR)f \neq 0$ (using $\varepsilon A \varepsilon' \neq 0$). Thus, e is a split idempotent in R, while its complementary idempotent f is *not* split. More explicitly, we can check that the Peirce corner

$$eRe = \begin{pmatrix} \varepsilon' A \varepsilon' & 0 \\ 0 & 0 \end{pmatrix} \text{ has the complement } C_e = \begin{pmatrix} \varepsilon A & A \\ 0 & A \end{pmatrix},$$

[5] Alternatively, $C_e = eRf \oplus fRf = Rf$ and $C_f = eRe \oplus eRf = eR$ are both ideals by the display $(*)$ prior to (2.11), which gives the same conclusions.

which is an ideal since εA is an ideal in A. On the other hand, the Peirce corner

$$fRf = \begin{pmatrix} \varepsilon A \varepsilon & \varepsilon A \\ 0 & A \end{pmatrix} \text{ has the complement } C_f = \begin{pmatrix} A\varepsilon' & \varepsilon' A \\ 0 & 0 \end{pmatrix},$$

which is not an ideal since $\varepsilon' A$ is not an ideal in A.

For a more concrete construction, let $A = \mathbb{T}_2(k)$ be the ring of 2×2 upper triangular matrices over a nonzero ring k, and let $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\varepsilon' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in A. Here, indeed, $\varepsilon' A \varepsilon = 0 \neq \varepsilon A \varepsilon'$. The construction above yields the ring

$$R = \{\, (a_{ij}) \in \mathbb{T}_4(k) : \ a_{23} = 0\} \subseteq \mathbb{T}_4(k), \tag{4.12}$$

with the complementary idempotents $e = e_{22}$ and $f = e_{11}+e_{33}+e_{44}$, where $\{e_{ij}\}$ are the matrix units. Here, the split corner eRe is just $k \cdot e_{22}$, with the ideal complement

$$C_e = \{\, (a_{ij}) \in \mathbb{T}_4(k) : \ a_{22} = a_{23} = 0\} \subseteq \mathbb{T}_4(k). \tag{4.13}$$

On the other hand, the nonsplit corner fRf is

$$\begin{pmatrix} k & 0 & k & k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix}, \quad \text{with the (non-ideal) complement } C_f = \begin{pmatrix} 0 & k & 0 & 0 \\ 0 & k & 0 & k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.14}$$

(4.15) Example. A suitable modification of the construction above can be used to produce complementary idempotents e, f (in a new ring R) that are *both* split, but not 1-sided semicentral. For $k \neq 0$ as above, let

$$R = \{\, (a_{ij}) \in \mathbb{T}_4(k) : \ a_{12} = a_{23} = a_{34} = 0\} \subseteq \mathbb{T}_4(k), \tag{4.16}$$

and take $e = e_{11} + e_{44}$, $f = e_{22} + e_{33}$. Then $eRf = ke_{13} \neq 0$, and $fRe = ke_{24} \neq 0$, so e, f are not 1-sided semicentral. But here,

$$e(RfR)e = k \cdot e_{13}e_{24} = 0, \quad \text{and} \quad f(ReR)f = k \cdot e_{24}e_{13} = 0,$$

so e, f are both split. The Peirce corners

$$eRe = ke_{11} + ke_{14} + ke_{44}\,, \quad \text{and} \quad fRf = ke_{22} + ke_{33}$$

have, respectively, the *ideal* complements

$$C_e = ke_{13} + ke_{22} + ke_{24} + ke_{33}\,, \quad \text{and} \quad C_f = ke_{11} + ke_{13} + ke_{14} + ke_{24} + ke_{44}\,.$$

§5. Reduction of Corners, and Correspondence of Complements

In this section, we shall prove a number of results that will clarify the special roles played by Peirce corners and unital corners in the general theory of corner rings. Specifically, we shall see that *any corner of a ring* R *is a unital corner of a Peirce corner of* R*, and is also a Peirce corner of a unital corner of* R*.* The significance of this is that, in many cases, the consideration of corners can be reduced to the two cases of Peirce corners and unital corners. We recall from (2.17), however, that there are examples of corners that are neither Peirce corners nor unital corners.

We start with the theme that any corner is a unital corner of a Peirce corner. This is quite easy to see: if $S \prec R$, say with identity e_0, then by (2.3)(1), $S \prec_u R_{e_o} \prec_P R$. Indeed, $R_{e_o} = e_0 R e_0$ is the smallest Peirce corner of R containing S; we shall call it the *associated Peirce corner of* S. In general, by considering *any* Peirce corner R_e containing S, we have the following reduction result relating the complements of S in R and in R_e, which follows easily from (2.9).[6]

(5.1) Theorem. *Given* $R_e \supseteq S \prec R$*, let* $\mathcal{C}$ *be the set of complements for* S *in* R*, and* $\mathcal{C}'$ *be the set of complements for* S *in* R_e*. Then there is a natural one-one correspondence between* $\mathcal{C}$ *and* $\mathcal{C}'$*, defined by* $C \mapsto C \cap R_e$*, and* $C' \mapsto C' \oplus C_e$ *(where* C_e *denotes the Peirce complement of* R_e*, as defined in* (2.8)$'$*). In particular,* $S \prec_r R$ *iff* $S \prec_r R_e$*.*

The next step is to try to develop some criteria for the corner S above to split in R. Again, we try to make a reduction to the splitting of S in any given Peirce corner R_e containing S (for instance, the associated Peirce corner R_{e_o} of S). For this, we shall use (4.1) and (4.5) as our blueprints, and try to extend them from the case R_e to the case $S \prec R_e$. [In particular, if we choose $e = e_0$ (the identity of S), we'll be reduced to the case of the splitting of a *unital* corner.] We start by taking any complement C of S in R (that is, $C \in \mathcal{C}$) and computing the ideal $\langle C \rangle$ generated by C in R. Throughout, we let $f = 1 - e$; recall that $f \in C$ by (2.2).

(5.2) Proposition. *Let* $C' = C \cap R_e$ *so that* $C = C' \oplus C_e$ *(as in* (5.1)*). Then*

$$\langle C \rangle = \langle C' \rangle_e + RfR\,, \tag{5.3}$$

where $\langle C' \rangle_e$ *denotes the ideal generated by* C' *in the ring* R_e*. In particular,* C *is an ideal of* R *iff* C' *is an ideal of* R_e *containing* $e(RfR)e$*.*

Proof. Since $f \in C$ and $C' \subseteq C$, the inclusion "$\supseteq$" is clear in (5.3). For the reverse inclusion, let $I := \langle C' \rangle_e + RfR$, which contains $C' \oplus C_e = C$. Thus, equality holds in (5.3) if we can show that I is an *ideal* in R. By left/right symmetry, it suffices to show that $R \cdot I \subseteq I$. Since $R = S \oplus C$, this amounts to showing $S \cdot I \subseteq I$ and

[6] In particular, (5.1) can be applied to the associated Peirce corner R_{e_o} of S. In this case, there is a slight simplification: for $C \in \mathcal{C}$ in (5.1), the contraction $C \cap R_{e_o}$ can also be expressed as $e_0 C e_0$.

$C \cdot I \subseteq I$. The former is immediate, since RfR is an ideal in R, and $\langle C' \rangle_e$ is an ideal in $R_e \supseteq S$. For the latter, we need only show that $C \cdot \langle C' \rangle_e \subseteq I$. But this is clear since

$$C = C' \oplus C_e \subseteq R_e + RfR\,,$$

and we have $R_e \cdot \langle C' \rangle_e \subseteq \langle C' \rangle_e$, and $RfR \cdot \langle C' \rangle_e \subseteq RfR \subseteq I$.

If C is an ideal in R, then $C' = C \cap R_e$ is an ideal in R_e, and $f \in C$ implies that $RfR \subseteq C$, whence $e(RfR)e \subseteq C'$. Conversely, assume that C' is an ideal of R_e containing $e(RfR)e$. By (5.3) and (4.1), we have

$$\langle C \rangle = \langle C' \rangle_e + RfR = C' + e(RfR)e + C_e = C' + C_e = C,$$

so C is an ideal in R. $\square$

We are now ready to prove the following result on the characterization of split corners.

(5.4) Theorem (Split Corner Criteria). *Let $R_e \supseteq S \prec R$ as before, and let $f = 1 - e$. Then S splits in R iff it has an ideal complement in R_e containing $e(RfR)e$, iff $S \cap e(RfR)e = 0$ and the image of S under the map $S \to R_e/e(RfR)e$ is a split corner of the ring $R_e/e(RfR)e$. In particular, $S \prec_{rs} R$ iff $S \cap e(RfR)e = 0$ and the image of S under the map $S \to R_e/e(RfR)e$ is a rigid-split corner in $R_e/e(RfR)e$.*

Proof. It is sufficient to prove the first "iff" statement. If $S \prec_s R$, choose for it an *ideal* complement C. By (5.2), $e(RfR)e \subseteq C' := C \cap R_e$, and so

$$S \cap e(RfR)e \subseteq S \cap C' \subseteq S \cap C = 0.$$

Conversely, assume S has an *ideal* complement J in R_e containing $e(RfR)e$. Let $C := J \oplus C_e$, which is a complement for S in R. Since $C' := C \cap R_e = J \supseteq e(RfR)e$, (5.2) implies that C is an ideal in R, so we have $S \prec_s R$, as desired. $\square$

(5.5) Corollary. *If a Peirce corner R_e contains a nonzero split corner S of R, then $f = 1 - e$ is not a full idempotent in R.*

Proof. By (5.4), $S \cap e(RfR)e = 0$. If f was a full idempotent, this would give $0 = S \cap eRe = S$. $\square$

Remark. If $S \prec_s R_e$ and $R_e \prec_s R$, then (2.16)(2) implies that $S \prec_s R$. However, the converse is not true; namely, $S \prec_s R$ implies only $S \prec_s R_e$, but in general does not imply that $R_e \prec_s R$. We shall demonstrate this with an example below.

(5.6) Example. Let k is a field of characteristic $\neq 2$, and let σ be the k-algebra automorphism on $A = k\,[x]$ defined by $\sigma(x) = -x$. Let $R = A \oplus Ay$, which is made into a ring using the rules $y^2 = 1$, and $y\alpha = \sigma(\alpha)y$ for every $\alpha \in A$. (In other words, R is the quotient obtained from the skew polynomial ring $A\,[y;\,\sigma]$ by factoring out the ideal generated by y^2-1.) Let e, f be the complementary idempotents $(1+y)/2$ and $(1-y)/2$ in R. In the factor ring R/RfR, y is identified with 1, so every $\alpha \in A$ is identified with $\sigma(\alpha)$. From this, it is easy to see that $R/RfR \cong k\,[x]/(x) \cong k$. Thus, $e \notin RfR$ (in particular f is non-full), and the subring $S := k \cdot e \prec_s R$, with an ideal complement RfR. However, by an easy computation,

$$exfxe = \frac{1}{8}(1+y)x(1-y)x(1+y) = x^2(1+y)/2 = x^2 e \neq 0,$$

so $e(RfR)e \neq 0$. Thus, the associated Peirce corner R_e of S fails to split, although S itself splits. A similar calculation shows that the other Peirce corner R_f is also non-split.

We now introduce the last type of corners in this paper, which is a certain weakening of split corners.

(5.7) Definition. We say that a corner $S \prec R$ (with identity e) is *semisplit* in R (written $S \prec_{ss} R$) if S is split in its associated Peirce corner $R_e = eRe$. A split corner is always semisplit, though not conversely. For instance, any Peirce corner is always semisplit, but not necessarily split. (For unital corners, of course, "split" and "semisplit" are synonymous.)

Note that, in this definition, the identity of the corner $S \prec R$ is denoted by e; in other words, the earlier notation e_0 is now replaced simply by e. This will be more convenient since, in the following, we shall only work with the associated Peirce complement R_e of S (instead of any Peirce complement containing S). The following result offers a couple of easy criteria for semisplit corners.

(5.8) Theorem (Semisplit Corner Criteria). *For a corner $S \prec R$ with identity element e, the following are equivalent:*

(1) $S \prec_{ss} R$*;*

(2) *S has a complement C in R such that $R_e C \subseteq C$ and $CR_e \subseteq C$;*

(3) *S has a complement C in R such that $(Re)C \subseteq C$ and $C(eR) \subseteq C$.*

Proof. (1) $\Rightarrow$ (2). Take an ideal complement I for S in R_e. Then $C := I \oplus C_e$ is a complement for S in R. We have

$$R_e\, C \subseteq R_e\, I + R_e\, C_e \subseteq I + C_e = C,$$

and similarly, $CR_e \subseteq C$, as desired.

(2) $\Rightarrow$ (3). Suppose C exists as in (2). By (2.9), we have $C \supseteq C_e$. Therefore,

$$(Re)\, C = (R_e + fRe)\, C \subseteq R_e\, C + fR \subseteq C + C_e = C.$$

Similarly, we can check that $C(eR) \subseteq C$.

(3) $\Rightarrow$ (1). Suppose C exists as in (3). Its contraction $C \cap R_e = eCe$ is a complement of S in R. The hypotheses on C imply that $R_e(eCe) \subseteq e(ReC)e \subseteq eCe$, and similarly, $(eCe)R_e \subseteq eCe$. Thus, eCe is an *ideal* in R_e, showing that $S \prec_{ss} R$. $\square$

Next, we take up the second theme of this section, which is that of realizing an arbitrary corner ring as a Peirce corner of a unital corner. This requires some nontrivial work. We begin more generally with the following observation.

(5.9) Lemma. *Let S and T be subrings of R with identities e, f such that $ef = fe = 0$. Then*

(1) *$S + T \prec R$ iff $S \prec R$ and $T \prec R$;*

(2) *$S + T \prec_r R$ iff $S \prec_r R$ and $T \prec_r R$.*

Proof. First note that $ST = (Se)(fT) = 0$, and $TS = (Tf)(eS) = 0$. Second, $S \cap T = 0$, for, if $r \in S \cap T$, then $r = er = e(fr) = 0$. Therefore, $S' := S + T \subseteq R$ is a subring with identity $e + f$, and S and T are direct Peirce corners of S'. For convenience, we may identity S' with the ring direct product $S \times T$.

(1) The "only if " part follows from the transitivity of corners. For the "if" part, assume that $S, T \prec R$, say with complements C, D respectively. For any $t \in T$, we have $ete = e(ft)e = 0$, so (2.7) implies that $t \in C$. Thus, $T \subseteq C$, and hence $C = C' \oplus T$, where $C' := C \cap D$. Now we have

$$R = S \oplus C = S \oplus (T \oplus C') = S' \oplus C',$$

so C' will be a complement to S' if we can show that $S'C' \subseteq C'$ and $C'S' \subseteq C'$. By symmetry, it suffices to show the former, which can be reduced to showing that $SC' \subseteq C'$ and $TC' \subseteq C'$. These in turn will follow if we can show the following four inclusions:

$$SC \subseteq C, \quad SD \subseteq D, \quad TC \subseteq C, \quad \text{and} \quad TD \subseteq D.$$

The first and the fourth are given. For the third, let $t \in T$ and $c \in C$, Then $e(tc)e = e(ft)ce = 0 \Rightarrow tc \in C$ by (2.7). Similarly, for $s \in S$ and $d \in D$, $f(sd)f = f(es)df = 0$ implies $sd \in D$, again by (2.7). This checks that $S' \prec R$, as desired.

(2) For the "only if" part, assume that $S' \prec_r R$. Let C, D, and $C' := C \cap D$ be as in (1) above. Since C' is a complement of S', it is uniquely determined. But then the earlier equation $C = C' \oplus T$ shows that C is also uniquely determined. Therefore, $S \prec_r R$, and similarly, $T \prec_r R$. The "if" part will follow from (5.10)(1) below. $\square$

The last main result of this section is the following theorem, which is in some sense parallel to (5.1) and (5.4).

(5.10) Theorem. *Let S, T be corners of R with identities e, f that are orthogonal idempotents. By* (5.9), *we have $S' := S \times T \prec R$. If T is rigid in R, with a (unique) complement D, the following conclusions hold:*

(1) *there is a one-one correspondence between $\mathcal{C}$, the set of complements for S in R, and $\mathcal{C}'$, the set of complements for S' in R, given by*

$$\alpha(C) \mapsto C \cap D \quad \textit{for} \;\; C \in \mathcal{C}, \quad \textit{and} \quad \beta(C') = T \oplus C' \quad \textit{for} \;\; C' \in \mathcal{C}'.$$

In particular, $S \prec_r R$ (and $T \prec_r R$) $\Longrightarrow$ $S' \prec_r R$.

(2) *$S' \prec_s R$ iff $S \prec_s R$ and $T \prec_s R$. In this case, under the one-one correspondence in* (1), *$C \in \mathcal{C}$ is an ideal in R iff the corresponding $C' \in \mathcal{C}'$ is an ideal in R; in particular, $S \prec_{rs} R$ iff $S' \prec_{rs} R$.*

Proof. (1) For $C \in \mathcal{C}$, we have shown in the proof of (5.9) that $C \cap D \in \mathcal{C}'$, so $\alpha(C) \in \mathcal{C}'$. On the other hand, for $C' \in \mathcal{C}'$, we have

$$R = S' \oplus C' = S \oplus (T \oplus C').$$

Also, $S \cdot (T \oplus C') = ST + SC' = SC' \subseteq S'C' \subseteq C'$, and similarly, $(T \oplus C')S \subseteq C'$. Thus, $\beta(C') := T \oplus C'$ is a complement to S; that is, $\beta(C') \in \mathcal{C}$. Now, for $C \in \mathcal{C}$, the proof of (5.9) gives

$$\beta(\alpha(C)) = \beta(C \cap D) = (C \cap D) \oplus T = C,$$

so $\beta \circ \alpha$ is the identity on $\mathcal{C}$. Finally, consider any $C' \in \mathcal{C}'$. Since $T \subseteq S'$ are both corners of R, C' can be "enlarged" into a complement for T (by (2.5)). Thus, the rigidity assumption on T forces $C' \subseteq D$. Therefore,

$$\alpha(\beta(C')) = \alpha(T \oplus C') = (T \oplus C') \cap D \supseteq C'.$$

Since C' and $\alpha(\beta(C'))$ are *both* complements of S', this implies that $\alpha(\beta(C')) = C'$, so $\alpha \circ \beta$ is also the identity on $\mathcal{C}'$. We have thus shown that α and β are mutually inverse one-one correspondences between $\mathcal{C}$ and $\mathcal{C}'$. This, of course, implies the last statement in (1).

(2) Assume that $S' \prec_s R$. Since $S' = S \times T$, (2.16)(2) implies that $S \prec_s R$ and $T \prec_s R$. In fact, the proof of (2.16)(2) shows that, for any *ideal* complement $C' \in \mathcal{C}'$, $\beta(C') = C' \oplus T$ is an ideal complement to S. Conversely, assume that $S \prec_s R$ and $T \prec_s R$, and consider any *ideal* complement $C \in \mathcal{C}$ (for S in R). Since T is rigid, $T \prec_s R$ implies that D is an ideal of R. Then $\alpha(C) = C \cap D$ is the intersection of two ideals, and is thus also an ideal. (In particular, $S' \prec_s R$.) This proves the one-one correspondence between the *ideal* complements in $\mathcal{C}$ and those in $\mathcal{C}'$, which, of course, also gives the last conclusion of (2). $\square$

(5.11) Corollary. *If $e_1, \ldots, e_n$ are mutually orthogonal idempotents in R, then $R_{e_1} \times \cdots \times R_{e_n}$ is a rigid corner in R. Its unique complement is $C_e \oplus \bigoplus_{i \neq j} e_i R e_j$, where $e := e_1 + \cdots + e_n$.*

Proof. Since each R_{e_i} is rigid, the rigidity of $R_{e_1} \times \cdots \times R_{e_n}$ follows from the last conclusion of (5.10)(1), plus induction on n. The computation of the (unique) complement of $R_{e_1} \times \cdots \times R_{e_n}$ is left to the reader. (Note that (3.5) is a special case of the present result.) $\square$

To see how Theorem (5.10) applies to our "second theme" (of realizing a corner as a Peirce corner of a unital corner), let us start with *any* corner $S \prec R$, with identity e. For the complementary idempotent $f := 1 - e$, we have $ef = fe = 0$, so the (rigid) Peirce corner $T := R_f = fRf$ satisfies the hypotheses of (5.9) and (5.10). Since $e + f = 1$, (5.9) implies that $S' := S + T = S \times R_f$ is a *unital* corner of R, and so S is a (direct) Peirce corner of this unital corner. The map $S \mapsto S \times R_f$ is a canonical "suspension process" that produces a unital corner from an arbitrary corner. As we shall see from the sequel of this paper [La$_4$], this suspension process is very useful in analyzing the behavior of the arbitrary corner S. To summarize, let us restate our main conclusions (from (5.10)) about $S \times R_f$, with the appropriate amendments in the present special case.

(5.12) Corollary. *For any corner $S \prec R$ with identity e, the "suspension" $S' := S \times R_f$ (for $f = 1 - e$) has the following properties:*

(0) *S' is a unital corner of R, containing S as a direct Peirce corner;*

(1) *the complements of S and those of S' are in one-one correspondence, with*

$$C \text{ (complement of } S) \mapsto C \cap C_f = eCe \oplus eRf \oplus fRe, \tag{5.13}$$

and C' (complement of S') $\mapsto C' \oplus R_f$. In particular, $S \prec_r R$ iff $S' \prec_r R$; and

(2) *$S' \prec_s R$ iff $S \prec_s R$ and f is a split idempotent. In this case, under the one-one correspondence in* (1), *ideal complements of S correspond to ideal complements of S'; in particular, $S \prec_{rs} R$ iff $S' \prec_{rs} R$.*

Proof. In (2) of (5.10), the condition that $T = R_f$ be split translates here into the splitting of the idempotent f in (2) above. Besides this, the only other point that requires an explanation is (5.13). Here, C_f is the Peirce complement of the idempotent f; that is, $C_f = eRf \oplus fRe \oplus fRf$. By (5.1), $C = (C \cap R_e) \oplus C_e = eCe \oplus C_e$. Therefore,

$$C \cap C_f = (eCe \oplus C_e) \cap C_f = eCe \oplus eRf \oplus fRe\,,$$

as claimed in (5.13). $\square$

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