Clean Matrices and Unit-Regular Matrices

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Abstract

An element in a ring R is said to be clean (resp. unit-regular) if it is the sum (resp. product) of an idempotent element and an invertible element. If all elements in R are unit-regular, it is known that all elements in R are clean. In this note, we show that a *single* unit-regular element in a ring need not be clean. More generally, a criterion is given for a matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ to be clean in a matrix ring $\mathbb{M}_2(K)$ over any commutative ring K. For $K = \mathbb{Z}$, this criterion shows, for instance, that the unit-regular matrix $\begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix}$ is not clean. Also, this turns out to be the "smallest" such example.

§1. Introduction

The notion of a unit-regular element was introduced by G. Ehrlich. According to [Eh], an element x in a ring R is *unit-regular* if $x = xux$ for some $u \in \mathrm{U}(R)$ (the group of units of R). It is easy to see that x is unit-regular iff x is an idempotent times a unit, iff x is a unit times an idempotent [La_2: Ex. (4.14B)]. As their name suggests, unit-regular elements are regular (in the sense of J. von Neumann). Ehrlich called a ring unit-regular if all of its elements are unit-regular. Rings of this kind have been extensively studied in the literature on von Neumann regular rings; see, e.g. [Go: Sec. 4].

In parallel, clean elements in rings were introduced by W. K. Nicholson. In [Ni_1], a ring element $x \in R$ is said to be *clean* if x is the sum of an idempotent and a unit in R. If all elements in R are clean, Nicholson called R a clean ring. Such rings are of interest since they constitute a subclass of the so-called *exchange rings* in the theory of noncommutative rings.

The relationship between cleanness and unit-regularity seems to be rather subtle. In [CY], or more correctly, in [CK], Camillo, Yu, and Khurana showed that *any unit-regular ring is clean.* This answered a question of Nicholson, but it does not say whether a *single* unit-regular element in a ring R need to be clean. In general, if an element $x \in R$ has the form eu where e is an idempotent and u is a unit *commuting with* e, then, writing $f = 1 - e$, we see that $x = f + (eu - f)$ is clean since f is an idempotent, and $eu - f$ is a unit (with inverse $eu^{-1} - f$). This shows that, in any ring in which idempotents are central (e.g. a reduced ring, a local ring, or a commutative ring), any unit-regular element is indeed clean.[1] More generally, in [Ni$_2$: Theorem 1], Nicholson has shown that, if $x \in R$ is such that some power x^n $(n \geq 1)$ has a factorization $eu = ue$ with $e = e^2$ and $u \in \mathrm{U}(R)$, then x is clean. This theorem implies that any strongly π-regular ring[2] is clean; in particular, any right artinian ring (e.g. finite ring) is clean.

The initial goal of this work was to show that, *in a noncommutative ring, unit-regular elements need not be clean.* A natural place to look for examples is the family of various kinds of matrix rings over a commutative ring K. Our first attempt working with the ring $\mathbb{T}_n(K)$ of upper triangular matrices over K did not bring about the desired examples, as it turned out, curiously enough, that *unit-regular elements are always clean in* $\mathbb{T}_n(K)$. Then, moving on to the case of full matrix rings $\mathbb{M}_n(K)$, we found our first examples from matrices of the form $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ (over suitable commutative rings K). Here, we solve the problem at hand by proving a general criterion (in Theorem 3.2) for the above matrix A to be clean in the ring $\mathbb{M}_2(K)$. As a consequence of this criterion, we show in (3.12) that $\begin{pmatrix} 1+xy & x^2 \\ 0 & 0 \end{pmatrix}$ (a derivative of the Cohn matrix in [Co]) is unit-regular but not clean over $K = k[x, y]$ for any commutative domain k. Specializing the cleanness criterion to the case $K = \mathbb{Z}$, we also obtain in §4 an algorithmically very simple method (Theorem 4.7) for deciding the cleanness of matrices of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ over $\mathbb{Z}$. In particular, we see that the choices $(a, b) = (12, 5)$, $(13, 5)$, $(12, 7)$, etc. lead to 2×2 unit-regular matrices over $\mathbb{Z}$ that are *not* clean. As it turns out, these are the "smallest" such examples.

As a by-product of this work, we obtain the most general form of a 2×2 idempotent matrix over a commutative ring with a prescribed (idempotent) determinant that has a unimodular second row. This result constitutes Theorem 5.1 below.

§2. Preliminary Results

In this section, we assemble a few results that will be needed in the subsequent sections. Throughout this paper, K denotes a commutative ring, and R denotes the 2×2 matrix ring $\mathbb{M}_2(K)$ over K.

To begin with, we determine all unit-regular matrices in R with a zero second row. This turned out to be fairly easy.

Proposition 2.1. *A matrix* $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ *is unit-regular in* $R = \mathbb{M}_2(K)$ *iff there exists an idempotent* $e \in K$ *and a unimodular row* $(a', b') \in K^2$ *such that* $(a, b) = e\,(a', b')$. *In particular, if* K *is connected (that is,* K *has only trivial idempotents), then the*

[1] On the other hand, the example $2 \in \mathbb{Z}$ shows that a clean element in a commutative domain need not be unit-regular.

[2] A ring R is *strongly π-regular* if, for any $x \in R$, $x^n \in x^{n+1}R$ for some n.

only nonzero unit-regular matrices in R *with a zero second row are* $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ *with* (a,b) *unimodular.*

Proof. If (a,b) has the form $e\,(a',b')$ as above, then $a'y - b'x = 1$ for some $x, y \in K$, and the equation

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a' & b' \\ x & y \end{pmatrix}$$

shows that A is unit-regular. Conversely, assume that A is unit-regular; say $A = EU$, where $E = E^2$ and $U \in \mathrm{GL}_2(K)$. Let $E = \begin{pmatrix} e & r \\ s & t \end{pmatrix}$, and $U = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. Then $(s,t) = (0,0) \cdot U^{-1} = (0,0)$, and so $E = E^2 = \begin{pmatrix} e^2 & er \\ 0 & 0 \end{pmatrix}$ shows that $e = e^2$, and $r = er$. Therefore, $(a,b) = e\,(a',b')$, where $(a',b') = (w,x) + r\,(y,z)$ is a unimodular row since (w,x) and (y,z) are the rows of a matrix in $\mathrm{GL}_2(K)$. $\square$

The problem of deciding the cleanness of $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is considerably harder. Naturally, the clean elements of K should play a key role in this problem. Here, we first dispose of the easy case where a itself is a clean element in K.

Proposition 2.2. *If* $a \in K$ *is clean, then for any* $b \in K$, $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ *is clean in* R.

Proof. If $a = e + u$ where $e = e^2$ and $u \in \mathrm{U}(R)$, then $A = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} u & b \\ 0 & -1 \end{pmatrix}$ shows that A is clean.[3] $\square$

Corollary 2.3. *let* K *be connected. If* $a \in K \setminus \{0\}$ *is clean but not a unit in* K*, then* $\begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}$ *is clean but not unit-regular in* $R = \mathbb{M}_2(K)$.

As was pointed out to us by Professor P. Ara, this Corollary is not true if K is not assumed to be connected. A counterexample is given by a (commutative) von Neumann regular ring K that is not a field. In this case, it is known that K and $\mathbb{M}_2(K)$ are both unit-regular rings, so the last conclusion of (2.3) cannot possibly hold.

We shall now introduce a matrix that will be crucial for the work in the rest of the paper. For any three elements $e, x, k \in K$ such that $e = e^2$ and $ex = 0$, we define the matrix

$$E(e,x,k) := \begin{pmatrix} e - kx & ke - k(kx+1) \\ x & kx+1 \end{pmatrix} \in R. \tag{2.4}$$

The basic properties of $E(e,x,k)$ are summarized as follows.

[3] A somewhat more sophisticated version of this result will be given in (3.4) below.

Proposition 2.5. *$E := E(e, x, k)$ is an idempotent matrix over K with* $\operatorname{tr}(E) = e+1$ *and* $\det(E) = e$.

Proof. The trace equation is clear, and by direct computation:

$$\det(E) = e(kx+1) - kx(kx+1) - xke + xk(kx+1) = e.$$

Furthermore, using the equation $ex = 0$, we get $e \cdot E = e \cdot I_2$. Thus, by the Cayley-Hamilton Theorem,

$$E^2 = \operatorname{tr}(E) \cdot E - \det(E) \cdot I_2 = (e+1)E - eE = E,$$

as desired. $\square$

For more information about $E(e, x, k)$, we might point out that, if we let

$$E_1 := \begin{pmatrix} e & ke \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad E_2 := \begin{pmatrix} -kx & -k(kx+1) \\ x & kx+1 \end{pmatrix}, \tag{2.6}$$

then $E = E_1 + E_2$, and (as we can easily check) E_1, E_2 are orthogonal idempotents in R. This shows that, if we think of $E(e, x, k)$ as an idempotent linear operator acting on the right of K^2, then its cokernel is isomorphic to the projective module K/eK. A strong uniqueness property for $E(e, x, k)$ will be proved in Theorem 5.1 below.

The following proposition will only be needed for 2×2 matrices, but we shall prove it for the $n \times n$ case since this does not call for any additional work.

Proposition 2.7. *Let $E = (a_{ij})$ be any $n \times n$ idempotent matrix over K with determinant e. Then $e^2 = e$, and $ea_{ij} = \delta_{ij} e$ (where $\{\delta_{ij}\}$ are the Kronecker deltas). If the last row of E happens to be unimodular, then* $a_{nn} \equiv 1 \pmod{a_{n1}K + \cdots + a_{n,n-1}K}$.

Proof. First, we have $e = \det(E) = \det(E^2) = (\det(E))^2 = e^2$. Let $f = 1 - e$ be the complementary idempotent of e. Over the factor ring K/fK, $\overline{E}$ has determinant $\overline{1}$, and is thus invertible. But then $\overline{E} = \overline{E}^2$ implies that $\overline{E}$ is the identity matrix. This means that $a_{ii} \in 1 + fK$ for all i, and $a_{ij} \in fK$ for $i \neq j$. Multiplying these by e, we see that $ea_{ii} = e$ for all i, and $ea_{ij} = 0$ for $i \neq j$. If, in addition, the last row of E is unimodular, then, over the factor ring $K/(a_{n1}K + \cdots + a_{n,n-1}K)$, a_{nn} becomes a unit, and E becomes an (idempotent) block-upper triangular matrix. The latter implies that the image of a_{nn} in $K/(a_{n1}K + \cdots + a_{n,n-1}K)$ is an idempotent, and thus we must have $a_{nn} \equiv 1 \pmod{a_{n1}K + \cdots + a_{n,n-1}K}$. $\square$

§3. Criterion for Some Clean Matrices

Throughout this section, A denotes the matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. In order to solve the problem of deciding the cleanness of A, it will be convenient to introduce a small technical variation of the notion of cleanness.

Definition 3.1. Let e be a given idempotent in K. If a matrix $M \in \mathbb{M}_n(K)$ can be written in the form $E+U$, for some $E = E^2$ of determinant e and some $U \in \mathrm{GL}_n(K)$, we shall say that M is *e-clean.* In particular, for $n = 1$, the e-clean elements of $K = \mathbb{M}_1(K)$ are just those of the form $e+u$ where $u \in \mathrm{U}(K)$.

Since (by (2.7)) an idempotent matrix has an idempotent determinant, a matrix M is clean iff it is e-clean for some idempotent $e \in K$. Although this idempotent is not uniquely determined by M in general, the definition above helps us to "catalogue" the clean matrices in a fashion. In view of these remarks, we could then transform the problem of deciding the cleanness of matrices into the problem of deciding their e-cleanness for any given idempotent $e \in K$. As it turns out, it is indeed in this form that the cleanness problem admits a reasonable solution, at least for matrices of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. The precise statement is as follows.

Theorem 3.2. *Let e be a given idempotent in K. Then $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is e-clean iff there exist $x, y \in K$ with $ex = 0$ and $y \equiv 1 \pmod{xK}$ such that $ay - bx$ is e-clean.*

Proof. For the "if" part, write y in the form $kx+1$ with $ex = 0$, and let $ay-bx = e+u$, where $u \in \mathrm{U}(K)$. We can then form the idempotent matrix $E := E(e, x, k)$ in (2.4), with $\det(E) = e$ by (2.5). Letting $U := A - E$, we have

$$\det(U) = -ay + bx + \det(E) = -(e+u) + e = -u \in \mathrm{U}(K).$$

Thus, $U \in \mathrm{GL}_2(K)$, and $A = E + U$ shows that A is e-clean.

For the "only if" part, assume that $A = E + U$, with $E = E^2 = \begin{pmatrix} p & q \\ x & y \end{pmatrix}$ of determinant e, and $U \in \mathrm{GL}_2(K)$. Since $U = \begin{pmatrix} a-p & b-q \\ -x & -y \end{pmatrix}$, we have $xK + yK = K$. Thus, by (2.7), we have $ex = 0$, and $y \equiv 1 \pmod{xK}$. Now let $u := -\det(U) \in \mathrm{U}(K)$. Then

$$-u = \det \begin{pmatrix} a-p & b-q \\ -x & -y \end{pmatrix} = -ay + bx + e,$$

so $ay - bx = e + u \in K$ is e-clean, as desired. $\square$

Corollary 3.3. *Let $k \in K$ and $v \in \mathrm{U}(K)$. For any idempotent $e \in K$, $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is e-clean iff $A' = \begin{pmatrix} a & vb+ka \\ 0 & 0 \end{pmatrix}$ is e-clean.*

Proof. It suffices to prove the "only if" part. Assuming A is e-clean, there exist (by Theorem 3.2) $x, y \in K$ such that

$$ex = 0, \quad y \equiv 1 \pmod{xK}, \quad \text{and} \quad ay - bx \text{ is } e\text{-clean}.$$

Let $b' := vb + ka$. Then $ay - bx = ay - v^{-1}(b' - ka)x = ay_1 - b'x_1$ is e-clean for $x_1 = v^{-1}x$ and $y_1 := y + v^{-1}kx$. Since $ex_1 = ev^{-1}x = 0$ and $y_1 \equiv y \equiv 1$ modulo the ideal $xK = x_1K$, A' is e-clean again by (3.2). $\square$

Remarks. (1) We could have also proved (3.3) by observing that $A' = V^{-1}AV$, where $V = \begin{pmatrix} 1 & k \\ 0 & v \end{pmatrix} \in \mathrm{GL}_2(K)$ (since the conjugate of an e-clean matrix remains e-clean). However, the proof given above is a good illustration of the use of (3.2).
(2) The roles of a and b in Theorem 3.2 are by no means symmetrical. Thus, the conclusion of (3.3) applies to A' only, and not to (for instance) $A'' = \begin{pmatrix} a + kb & b \\ 0 & 0 \end{pmatrix}$. This can be seen easily, for instance, from (3.5)(1) below, or from the many examples in §4.

Corollary 3.4. *Let $r \in K$ be e-clean, where e is an idempotent of K. Then for any b, $k \in K$, the matrix $B = \begin{pmatrix} (1-e)kb + r & b \\ 0 & 0 \end{pmatrix}$ is e-clean.*

Proof. For $a := (1-e)kb + r$, $y = 1$, and $x = (1-e)k$, we have

$$ay - bx = (1-e)kb + r - b(1-e)k = r.$$

Since $ex = e(1-e)k = 0$, $y \equiv 1 \pmod{xK}$, and r is e-clean, (3.2) implies that the matrix B is e-clean. (Note that (2.2) is essentially the special case of this result for $k = 0$.) $\square$

For later use let us record the following more explicit version of (3.2) corresponding to the case of trivial idempotents $e = 0, 1$.

Corollary 3.5. (1) *A is 1-clean iff $a \in 1 + \mathrm{U}(K)$.* (2) *A is 0-clean iff there exist $x_0, y_0 \in K$ such that $ay_0 - bx_0 = 1$ and $y_0 + x_0K$ contains a unit of K. (In this case, A is also unit-regular, according to* (2.1)*.)*

Proof. (1) follows by letting $e = 1$ in (3.2) and noting that this implies that $x = 0$ and $y = 1$. With these uniquely determined values of x and y, the 1-clean criterion for A boils down to $a \in 1 + \mathrm{U}(K)$. (This criterion is easy to see in any case, since the only idempotent matrix of determinant 1 is the identity matrix.)

For (2), let $e = 0$, in which case the condition $ex = 0$ is automatic. The 0-clean criterion for A then requires the existence of $x, y \in K$ with $y \equiv 1 \pmod{xK}$ such that $u := ay - bx \in \mathrm{U}(K)$. Upon dividing everything by u, this transforms into the existence of x_0, $y_0 \in K$ such that $ay_0 - bx_0 = 1$ and $(y_0 + x_0K) \cap \mathrm{U}(K) \neq \emptyset$. $\square$

In the case where K is connected, a clean matrix is either 0-clean or 1-clean. Here, the cleanness criterion for A can be more conveniently stated as follows (via (3.5)).

Corollary 3.6. *Let K be a connected (commutative) ring.* (1) *If $a \in 1 + \mathrm{U}(K)$, then A is always clean.* (2) *If otherwise, then A is clean iff there exist x_0, $y_0 \in K$ such that $ay_0 - bx_0 = 1$ and $y_0 + x_0K$ contains a unit of K. (In this case, A is also unit-regular.)*

Another consequence of (3.3), (3.4), and (3.5)(2) useful for the next section is the following (for any commutative ring K).

Corollary 3.7. *Let $a, b, k \in K$, and u, $v \in \mathrm{U}(K)$.*
(1) *If* $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ *is* 0*-clean, so is* $\begin{pmatrix} ua & vb+ka \\ 0 & 0 \end{pmatrix}$.
(2) *The matrix* $\begin{pmatrix} kb+u & b \\ 0 & 0 \end{pmatrix}$ *is always* 0*-clean.*

Proof. (1) If $u = 1$, this is covered by (3.3) (even in the e-clean case). Thus, it only remains to make the passage from A to $\begin{pmatrix} ua & b \\ 0 & 0 \end{pmatrix}$. This can be done (albeit only in the 0-clean case) by rewriting the equation $ay_0 - bx_0 = 1$ in (3.5)(2) in the form $au(u^{-1}y_0) - bx_0 = 1$, and noting that

$$u^{-1}y_0 + x_0K = u^{-1}(y_0 + x_0uK) = u^{-1}(y_0 + x_0K).$$

Finally, (2) follows from (3.4) by letting $e = 0$. $\square$

There are also a few interesting connections between (3.5) and notions in algebraic K-theory that are worth mentioning. Recall that a ring K is said to have *stable range* 1 if, for any x, $y \in K$, $yK + xK = K$ implies that $y + xK$ contains a unit of K (see [La$_1$: §20]). In this case, the 0-clean criterion in (3.5)(2) above can be further simplified as follows.

Corollary 3.8. *If K has stable range* 1*, A is* 0*-clean iff its first row (a, b) is unimodular over K.*

A second connection between 0-cleanness and the K_1-functor in algebraic K-theory is provided by $\mathrm{E}_n(K)$, the subgroup of $\mathrm{SL}_n(K)$ generated by the elementary matrices. According to a theorem of Suslin [GM: p. 14], $\mathrm{E}_n(K)$ is a normal subgroup of $\mathrm{SL}_n(K)$ whenever $n \geq 3$ (and K is a commutative ring). For any unimodular row $(a,\, b)$, the *Mennicke symbol* $[\,a,\, b\,]$ is defined to be the class in the factor group $\mathrm{SL}_3(K)/\mathrm{E}_3(K)$ given by any matrix $\begin{pmatrix} a & b & 0 \\ x & y & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where $ay - bx = 1$. (See [GM: p. 43].) The following consequence of (3.5)(2) gives some interesting necessary conditions for 0-cleanness.

Proposition 3.9. *If* $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ *is* 0*-clean, then*
(1) *$(a,\, b)$ can be brought to $(1, 0)$ by a finite sequence of elementary transformations*;
(2) *$(a,\, b)$ is completable to a matrix in $\mathrm{E}_2(K)$; and*
(3) *any completion of (a, b) to a matrix in $\mathrm{SL}_2(K)$ belongs to $\mathrm{E}_2(K)$.*
In particular, the Mennicke symbol $[\,a,\, b\,] \in \mathrm{SL}_3(K)/\mathrm{E}_3(K)$ is trivial.

Proof. For any unimodular row (a, b), it is easy to show that the three properties (1), (2), (3) above are equivalent. If A is 0-clean, (a, b) is unimodular by (3.5)(2), so we are done if we can prove (2). By (3.5)(2), there exist $x_0,\, y_0 \in K$ such that $ay_0 - bx_0 = 1$ and $u := y_0 + x_0 k$ is a unit for some $k \in K$. Then $B := \begin{pmatrix} a & b \\ x_0 & y_0 \end{pmatrix} \in \mathrm{SL}_2(K)$. For $C_1 = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, we have $BC_1 = \begin{pmatrix} a & b+ka \\ x_0 & u \end{pmatrix}$. Further right-multiplying this by $C_2 = \begin{pmatrix} 1 & 0 \\ -x_0u^{-1} & 1 \end{pmatrix}$, we get a matrix of the form $C_3 = \begin{pmatrix} c & d \\ 0 & u \end{pmatrix}$. Since this matrix remains in $\mathrm{SL}_2(K)$, we have $c = u^{-1}$. Now Whitehead's Lemma (as in [LS: p. 344]) implies that $C_3 \in \mathrm{E}_2(K)$. Thus, $B = C_3C_2^{-1}C_1^{-1} \in \mathrm{E}_2(K)$, proving (2). In particular, we have $\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{E}_3(K)$, so the Mennicke symbol $[\,a,\, b\,]$ is trivial. $\square$

Corollary 3.10. *Suppose there exists a matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K) \setminus \mathrm{E}_2(K)$, *where* K *is a connected ring. If* $a \notin 1 + \mathrm{U}(K)$, then the matrix $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ *is unit-regular but not clean. In particular, this conclusion on* A *always holds for any unimodular row* (a, b) *with* $a \notin 1 + \mathrm{U}(K)$ *that gives a nontrivial Mennicke symbol* $[\,a,\, b\,]$.

Proof. By (3.9), A is not 0-clean, and by (3.5)(1), A is not 1-clean. Since K is connected, A is not clean. But A is unit-regular by (2.1). $\square$

With (3.10) at our disposal (and assuming some results in the literature), we can now give some quick examples of 2×2 unit-regular matrices that are not clean.

Example 3.11. Let K be the real coordinate ring of the circle S^1; that is, $K = \mathbb{R}\,[a, b]$, with the relation $a^2 + b^2 = 1$. It is known that, for the matrix $B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{SL}_2(K)$, the suspension $\begin{pmatrix} B & 0 \\ 0 & I_n \end{pmatrix}$ is not in $\mathrm{E}_{n+2}(K)$ for any n; see, e.g. [LS: p. 345]. In particular, the Mennicke symbol $[\,a,\, b\,]$ is nontrivial. Since K is connected and $a \notin 1 + \mathrm{U}(K)$, the last part of (3.10) implies that $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is unit-regular but not clean. Note that A here is pretty close to being a "generic" counterexample. The "generic" one would be $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ over the commutative $\mathbb{R}$-algebra $\mathbb{R}\,[a, b, c, d]$ with the relation $ad - bc = 1$. In retrospect, we now know that this is *indeed* a counterexample!

Example 3.12. We can also give an example where the Mennicke symbol is trivial. Let K be the polynomial ring $k\,[x, y]$, where k is any commutative domain. According to Cohn [Co] and Gupta-Murthy [GM: p. 16], the Mennicke symbol $[\,1 + xy,\, x^2\,]$ is trivial, although the "Cohn matrix"

$$B = \begin{pmatrix} 1 + xy & x^2 \\ -y^2 & 1 - xy \end{pmatrix} \in \mathrm{SL}_2(K) \tag{3.13}$$

does not lie in $\mathrm{E}_2(K)$. Using (3.10) again, we see as in the last example that the matrix $A = \begin{pmatrix} 1+xy & x^2 \\ 0 & 0 \end{pmatrix}$ is unit-regular but not clean in $\mathbb{M}_2(K)$.

Finally, we note that the converse of (3.9) is false in general; that is, even if (a, b) is the first row of a matrix in $\mathrm{E}_2(K)$, the matrix $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ may still be non-clean. For instance, over the Euclidean domain $\mathbb{Z}$, we have $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{E}_2(\mathbb{Z})$, so *any* unimodular row (a, b) is completable to a matrix in $\mathrm{E}_2(\mathbb{Z})$. But a matrix such as $\begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix}$ turns out to be non-clean over $\mathbb{Z}$, as we shall see in (4.5) below.

§4. Clean and Non-Clean Matrices Over $\mathbb{Z}$

In this section, we shall specialize the results of §3 to the ring of integers; that is, we let $K = \mathbb{Z}$. In this case, it turns out that the criterion for the cleanness of a unit-regular matrix of the form $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ can be further simplified. In fact, the simplified criterion is in such a form that the cleanness of A can be quickly tested by carrying out the Euclidean algorithm. The results in this section were worked out in collaboration with Tom Dorsey and Alex Dugas, whose contributions are gratefully acknowledged here.

Throughout this section, A continues to denote the matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. In studying the cleanness of A over $\mathbb{Z}$, we are free to change the signs of a and b (by (3.7)), so we may assume that $a, b \geq 0$ whenever it is convenient to do so. We begin with the following useful observation on cleanness versus 0-cleanness in the case $K = \mathbb{Z}$.

Proposition 4.1. *The matrix A is 0-clean iff it is clean and $(a, b) \in \mathbb{Z}^2$ is unimodular.*

Proof. The "only if" part is clear from (3.5)(2). Conversely, assume that A is clean with (a, b) unimodular. If A is not 0-clean, then it must be 1-clean (since $\mathbb{Z}$ has only trivial idempotents). Therefore, by (3.5)(1), $a \in \{0, 2\}$. If $a = 0$, then $b = \pm 1$; in this case, the 0-clean criterion for A in (3.5)(2) is fulfilled by choosing (say) $y_0 = 1$ and $x_0 = \mp 1$. If $a = 2$, then $b = 2n + 1$ for some n; here, the 0-clean criterion in (3.5)(2) is fulfilled by choosing $y_0 = -n$ and $x_0 = -1$. □

In view of (2.1), there is no real loss if we restrict ourselves to the case where (a, b) is unimodular. Then "clean" becomes synonymous with "0-clean", according to (4.1). We start by giving a couple of numerical examples to illustrate the process of testing cleanness and computing "clean decompositions" by the constructive proof of (3.2).

Example 4.2. To test by (3.2) if $\begin{pmatrix} 9 & 7 \\ 0 & 0 \end{pmatrix}$ is clean (and hence 0-clean), we deal with the diophantine equations $9y - 7x = u \in \mathrm{U}(\mathbb{Z}) = \{\pm 1\}$. For $u = 1$, this has a solution $y = -3$, $x = -4$ satisfying $y \equiv 1 \pmod{x}$. Therefore, by Theorem 3.2 (and its proof),

A is 0-clean, with an idempotent summand $E = E(0, -4, 1)$. Thus, we arrive at a clean representation:

$$\begin{pmatrix} 9 & 7 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -4 & -3 \end{pmatrix} + \begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix}. \tag{4.3}$$

On the other hand, we could have also solved the equation $9y - 7x = u$ for $u = -1$ with $y = -4$ and $x = -5$ again satisfying $y \equiv 1 \pmod{x}$. This gives a different idempotent summand $E(0, -5, 1)$ for A, leading to a second explicit clean representation:

$$\begin{pmatrix} 9 & 7 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & -4 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}. \tag{4.4}$$

Example 4.5. Let us show that $A = \begin{pmatrix} 12 & 5 \\ 0 & 0 \end{pmatrix}$ is *not* clean (although A is unit-regular by (2.1)). Applying (3.5)(2), we want to show that, for any $(x_0, y_0) \in \mathbb{Z}^2$ solving $12y_0 - 5x_0 = 1$, $y_0 + x_0\mathbb{Z}$ contains neither 1 nor -1. Now the general solution for the diophantine equation $12y_0 - 5x_0 = 1$ is $y_0 = -2 + 5t$, $x_0 = -5 + 12t$ (with $t \in \mathbb{Z}$). Clearly, neither $-1 + 5t$ nor $-3 + 5t$ can be a multiple of $-5 + 12t$, for any $t \in \mathbb{Z}$. Therefore, the matrix A is *not* clean in $\mathbb{M}_2(\mathbb{Z})$. The same work applies to the matrix $\begin{pmatrix} 13 & 5 \\ 0 & 0 \end{pmatrix}$, and an application of (3.3) generates two more examples: $\begin{pmatrix} 12 & 7 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 13 & 8 \\ 0 & 0 \end{pmatrix}$.

Of course, for a given matrix A, the cleanness criterion in (3.5) is explicit enough to be checked by a computer. A program written by Tom Dorsey showed that the non-negative unimodular rows $(a, b) \in \mathbb{Z}_+^2$ leading to "dirty" unit-regular matrices are the following, listed in increasing values[4] of $a + b$:

$$\begin{array}{c} (12,5),\ (13,5),\ (12,7),\ (13,8),\ (17,5),\ (16,7),\ (18,5),\ (17,7), \\ (16,9),\ (18,7),\ (19,7),\ (17,10),\ (19,8),\ (22,5),\ (23,5),\ (12,17), \\ (17,12),\ (18,11),\ (20,9),\ (21,8),\ (19,11),\ (23,7),\ (12,19),\ \text{etc.} \end{array} \tag{$*$}$$

Thus, at least in the initial range $1 \leq a + b \leq 30$, examples of non-clean unit-regular matrices of the form A occur rather scarcely, and the examples mentioned in (4.5) correspond, in fact, to the four smallest choices for $s := a + b$, each occurring uniquely for the given row sum s.

The fact that all rows (a, b) listed in $(*)$ have $a \geq 12$ and $b \in \{5\} \cup [\,7, \infty)$ led us to the following general observations, which will be proved independently of the tabulation $(*)$ above.

Proposition 4.6. *Assume that the row $(a, b) \in \mathbb{Z}^2$ is unimodular. Then,*

[4] For rows (a, b) with the same row sum $a + b$, we list them in increasing values of a.

(1) *the matrix* A *is clean if* b *is congruent* $(\mathrm{mod}\ a)$ *to* ± 1, ± 2, ± 3, ± 4 *or* ± 6*; and*
(2) A *is always clean if* $|a| < 12$*, or if* $|a| + |b| \in [1,\ 16] \cup \{20\}$.

Proof. (1) Applying (3.3), and changing b to $-b$ if necessary, we may assume that b is one of 1, 2, 3, 4 or 6. We shall only treat the case $b = 6$, the other cases being completely similar. Since $\mathrm{U}(\mathbb{Z}/6\mathbb{Z}) = \{\pm 1\}$, we can write a in the form $6n \pm 1$. Then we can solve the equation $ay_0 - bx_0 = 1$ with $x_0 = \pm n$, $y_0 = \pm 1$ (so in particular $y_0 \equiv \pm 1 \ (\mathrm{mod}\ x_0)$). Therefore, by (3.5)(2), the matrix A is 0-clean.

(2) If $|a| < 12$, the hypothesis on b above is clearly satisfied, so A is always clean in this case. Next, assume that $|a| + |b| \in [1,\ 16] \cup \{20\}$. Changing signs if necessary, we may also assume that $b > 0$ and $a \geq 12$. If $a + b \leq 16$, then $b \leq 4$, so A is clean by (1). If $a + b = 20$ and $b > 4$, the only possibility is $(a, b) = (13, 7)$. But this is not in $(*)$ by (3.3) since $(13, 13 - 7) = (13, 6)$ is not in $(*)$ by (1) above. Thus, A is clean in all the cases claimed. $\square$

In the case $a = 12$ (resp. $a = 13$), the two cases *not* covered by the Proposition above are $b = 5,\ 7$ (resp. $b = 5,\ 8$). And indeed, these choices are responsible for the first four rows in the tabulation $(*)$.

The idea of using the transformation $(a, b) \mapsto (a,\ b - ka)$ in the proof of (4.6) also suggests the best way to analyze the general data in $(*)$. Note, for instance, that $(12, 17)$ is the first occurrence of a row in $(*)$ with $a < b$. But $(12, 17)$ could have been obtained from the first row $(12, 5)$ in $(*)$ by the transformation $(a, b) \mapsto (a,\ b + a)$. Thus, the presence of $(12, 5)$ "induces" that of $(12, 17)$. In general, if (a, b) is present in $(*)$ with $b = ka + r$ where $k \in \mathbb{Z}$ and $0 < r < a$, then (3.3) implies that (a, r) must be an "earlier" entry in $(*)$ (whose presence induces that of (a, b)). Once we are down to a row (a, r) with $0 < r < a$, we may also assume that $a \geq 2r$, for if otherwise, we may perform the transformations $(a, r) \mapsto (a,\ r - a) \mapsto (a,\ a - r)$ to achieve our goal.

Let us say that a unimodular row $(a, b) \in \mathbb{Z}^2$ is *reduced* if $|a| \geq 2\,|b|$. By the considerations in the paragraph above, any row in the tabulation $(*)$ can be brought in a canonical fashion to an earlier reduced row in $(*)$. Therefore, for the purposes of deciding which unimodular rows $(a, b) \in \mathbb{Z}_+^2$ are in $(*)$, *there is no loss of generality in working with the reduced rows* (a, b). Thus, the problem of deciding the cleanness of A will be solved in an algorithmically very simple fashion as soon as we prove the following result.

Theorem 4.7. *Let* $(a, b) \in \mathbb{Z}_+^2$ *be a reduced unimodular row. Then* A *is clean iff* $a \equiv \pm 1 \ (\mathrm{mod}\ b)$.

Proof. If $b = 0$, then $a = 1$, in which case both statements are true. Therefore, in the following, we may assume that $b > 0$.

The "if" part follows from (3.7)(2) (even without the reducedness assumption on (a, b)). For the "only if" part, assume that $a \not\equiv \pm 1 \ (\mathrm{mod}\ b)$. By the Euclidean algorithm, we can write $a = kb + r$, where $0 < r < b$ and $k \in \mathbb{Z}$. Since $a \geq 2b$, $k \geq 2$. Applying (3.5)(2), it suffices to check that, for any integers x, y solving

$$1 = ay - bx = (kb + r)y - bx, \tag{4.8}$$

we have $y \not\equiv \pm 1 \pmod{x}$. From (4.8), we have $ay \equiv ry \equiv 1 \pmod{b}$, so $a \not\equiv \pm 1 \pmod{b}$ implies that $y \not\equiv \pm 1 \pmod{b}$. In particular, $b \neq 1$ and $|y| \geq 2$. Using the equation $x = ky + (ry - 1)/b$, we argue in two cases.

Case 1. $y < 0$. Since $(ry - 1)/b < 0$, $x < ky \leq 2y < y < -1$. This clearly implies that $y \not\equiv \pm 1 \pmod{x}$.

Case 2. $y > 0$. Here, $(ry - 1)/b > 0$, so $x \geq 1 + ky \geq 1 + 2y$. Since $y \geq 2$, this again implies that $y \not\equiv \pm 1 \pmod{x}$, as desired. $\square$

While the reducedness assumption on (a, b) is not needed for the "if" part above, it is essential for the "only if" part. For instance, in (4.1), the matrix A with a (non-reduced) top row $(9, 7)$ is clean, but $a = 9$ is not congruent to $\pm 1 \pmod{b}$ for $b = 7$. In fact, the "only if" part in (4.7) is false even if we try to relax the reducedness assumption $b \leq a/2$ to $b < 3a/4$, as the example $(28, 19)$ shows.

The new method given in Theorem 4.7 for testing the cleanness of A (over the integers) is algorithmically very easy to implement since the decision procedure involves nothing more than the Euclidean algorithm. Let us illustrate this with a couple of examples.

Examples 4.9. To determine all (a, b) in $(*)$ with first entry $a = 36$, we may assume $(36, b)$ is unimodular and reduced. Thus, $b \in \{1, 5, 7, 11, 13, 17\}$. Among these choices of b, only the first three satisfy $36 \equiv \pm 1 \pmod{b}$. Therefore, by (4.7), the reduced rows in $(*)$ with $a = 36$ are $(36, 11)$, $(36, 13)$, and $(36, 17)$. Similarly, we can check, for instance, that a reduced row of the form $(69, b)$ occurs in $(*)$ iff

$$b \in \{8,\ 11,\ 13,\ 16,\ 19,\ 20,\ 22,\ 25,\ 26,\ 28,\ 29,\ 31,\ 32\}.$$

Theorem 4.7 can also be used in various ways to give more precise quantitative information about the rows in $(*)$ corresponding to unit-regular matrices A that are non-clean. Its first natural consequence is the following.

Corollary 4.10. *Let $b > 0$. Then there exists a row of the form (a, b) in the tabulation $(*)$ iff $b \notin \{1, 2, 3, 4, 6\}$.*

Proof. The "only if" part is a special case of (4.6)(1). For the converse, assume that $b \notin \{1, 2, 3, 4, 6\}$. It is well-known that in this case $\mathrm{U}(\mathbb{Z}/b\mathbb{Z}) \supsetneq \{\pm 1\}$. Thus, there is a positive integer $r < b$ relatively prime to b such that $r \not\equiv \pm 1 \pmod{b}$. Then for any integer $k \geq 2$, $(kb + r,\ b)$ is a reduced unimodular row, and the matrix A with first row $(kb + r,\ b)$ is non-clean by (4.7). Therefore, the row $(kb + r,\ b)$ occurs in the tabulation $(*)$ for every integer $k \geq 2$. $\square$

For a row $(a,b)\in\mathbb{Z}^2$, let $s=|a|+|b|$ denote its (absolute) row sum (also known as its "L^1-norm"). We saw in the tabulation $(*)$ (and proved by hand in (4.6)(2)) that the row sums $1,2,\ldots,16$ and 20 are "missing". A missing sum s means, of course, that the matrix A is clean whenever (a,b) is unimodular with $|a|+|b|=s$. An obvious question to ask is *whether there exist in* $(*)$ *other missing row sums beyond* 20. With the help of Alex Dugas and Michael Filaseta, we shall give a negative answer to this question below.

It all comes down to using (4.7) efficiently. If we use the proof of its corollary (4.10), we may start with any $b=2n+1\geq 5$, and take $r=k=2$ in that proof to generate a row $(2b+2,\,b)$ in $(*)$. This gives a row sum $3b+2=6n+5$ for any $n\geq 2$. Similar constructions would generate other infinite sequences of row sums. If we want to get *all* row sums (beyond 20) in one stroke, we may proceed as follows. By standard results in prime number theory,[5] there exists a bound n_0 such that, for any integer $n\geq n_0$, there is a prime p satisfying

$$\frac{n+1}{4}<p<\frac{n-1}{3}. \tag{4.11}$$

For such a prime p, we have $3p<n-1<n<n+1<4p$. Therefore, $n=3p+r$ for some $r\in[\,2,\,p-2]$. Taking $b=p$ and $a=2p+r$, we have $a\equiv r\not\equiv\pm 1 \pmod b$, and so by (4.7), the (reduced) unimodular row (a,b) must show up in the tabulation $(*)$, with row sum $a+b=3p+r=n$. This construction produces *all* row sums $n\geq n_0$, thus proving that there can only be finitely many "missing" row sums in $(*)$.

Now we are reduced to showing that there are no missing sums >20 in the range $[1,\,n_0)$. Of course this would be an easier problem only if we can name an explicit bound n_0! Fortunately, help is available from the classical number theory literature. In [Sch: Hilfssatz 1], Schur proved that, for any real number $x\geq 29$, there exists a prime p such that $x<p\leq 5x/4$. This can be applied to our situation since $5/4$ is (if only a little) less than $4/3$: letting $x=(n+1)/4$, we see that for $n\geq 115$, there is always a prime $p>(n+1)/4$ such that

$$p\leq\frac{5}{4}\cdot\frac{n+1}{4}=\frac{5(n+1)}{16}<\frac{n-1}{3}.$$

Thus, we may take $n_0=115$, although we certainly would not expect this to be the best bound. In fact, a quick check on Maple shows that the required prime p in (4.11) exists also for every n in the range $[\,53,\,115)$. On the other hand, if $n=52$, there is no prime in the open interval $((n+1)/4,\,(n-1)/3)=(13.25,\,17)$. Therefore, the best bound for our analytic problem is $n_0=53$: this was pointed out to us by M. Filaseta. Finally, a quick hand computation (or direct inspection of the computer data for $(*)$) shows that all sums from 21 to 52 are also "realized"! *Therefore, the missing sums in* $(*)$ *are precisely:* $1,2,\ldots,16$*, and* 20. Moreover, any sum s that is not one of these exceptional values can be realized by some (a,b) in $(*)$ with b a prime number.

[5] More details on this point will be given in the next paragraph.

The method above actually gives all missing sums for the *reduced* rows in $(*)$ as well. We may drop all non-reduced rows from $(*)$ to form a new list $(*)_{\text{red}}$, and ask for the missing sums herein:

$$(*)_{\text{red}} \qquad \begin{array}{c} (12,5),\ (13,5),\ (17,5),\ (16,7),\ (18,5),\ (17,7),\ (18,7), \\ (19,7),\ (19,8),\ (22,5),\ (23,5),\ (20,9),\ (21,8),\ (23,7),\ \text{etc.} \end{array}$$

Since we have dropped $(12,7)$ and $(13,8)$, the sums 19 and 21 are now also missing (along with 20). However, the construction in the last paragraph used only reduced rows, and it can be checked easily that the sums in the interval $[22, 52]$ can each be realized by reduced rows. We may, therefore, conclude that, *in the new list* $(*)_{\text{red}}$ *of reduced rows giving rise to non-clean matrices* A*, the missing sums are precisely* $1, 2, \ldots, 16$*, and* 19, 20, 21.

We close this section by pointing out that the examples of non-clean integral matrices obtained in this section naturally induce many examples of the same type over other rings. For instance, we can infer that $\begin{pmatrix} 1+xy & x^2 \\ 0 & 0 \end{pmatrix}$ is (unit-regular and) non-clean over $\mathbb{Z}[x, y]$ by specializing x to 3 and y to 5 (and noting that $(16, 9)$ is in $(*)$).[6] Similarly, since $(1+x)^2 = 1 + x(2+x)$, we see that $\begin{pmatrix} (1+x)^2 & x^2 \\ 0 & 0 \end{pmatrix}$ is unit-regular and non-clean over the polynomial ring $\mathbb{Z}[x]$.

§5. Idempotent Matrices with a Unimodular Row

In this final section, we return to the case of an arbitrary commutative base ring K. In §3, we were able to prove our main result Theorem (3.2) by making use of the matrices $E(e, x, k)$ defined in (2.4). However, we have not given any explanation as to how we arrived at these idempotent matrices. In this section, we shall prove a strong characterization theorem on the matrices $E(e, x, k)$, which, in particular, shows how they are constructed from three specific defining properties. We should, however, stress the fact that, while the matrices $E(e, x, k)$ seemed crucial for the proof of Theorem 3.2, the following characterization result on such matrices was not needed in that proof.

Theorem 5.1. *Let* e *be a given idempotent in* K*. Then the most general idempotent matrix* $E \in R = \mathbb{M}_2(K)$ *with a unimodular second row and with* $\det(E) = e$ *is* $E(e, x, k)$*, where* $x, k \in K$*, with* $ex = 0$*.*

Proof. If $ex = 0$, we have shown in (2.5) that $E(e, x, k)$ is idempotent, has determinant e, and its second row $(x,\ kx+1)$ is obviously unimodular. Conversely, let $E = \begin{pmatrix} p & q \\ x & y \end{pmatrix}$ be an idempotent matrix with $\det(E) = e$, and with (x, y) unimodular. By (2.7), we have

$$(5.2) \qquad\qquad eq = ex = 0, \quad \text{and} \quad ey = e,$$

[6] This, however, would not work over $k[x, y]$ where k is an arbitrary commutative domain. For this case, we have to go back to (3.12)!

and we can write y in the form k_0x+1 for some $k_0 \in K$. Consider the two linear equations

$$-qx+py=e, \quad \text{and} \quad -k_0x+y=1,$$

and let $k := \det\begin{pmatrix} -q & p \\ -k_0 & 1 \end{pmatrix}$. By Cramer's Rule, we have $kx = \det\begin{pmatrix} e & p \\ 1 & 1 \end{pmatrix} = e-p$, and $ky = \det\begin{pmatrix} -q & e \\ -k_0 & 1 \end{pmatrix} = -q+k_0e$. Therefore, $p = e-kx$, and $q = k_0e-ky$. Let $r := \mathrm{tr}(E)-1 \in K$. By the Cayley-Hamilton Theorem (and the fact that $E = E^2$), we have $r \cdot E = e \cdot I_2$, so $rx = 0$, and $ry = e$. This yields $e = r(k_0x+1) = r$. Therefore,

$$e = \mathrm{tr}(E)-1 = p+y-1 = e-kx+k_0x,$$

which implies that $kx = k_0x$. It follows that

$$y = kx+1, \quad \text{and} \quad q = k_0e-k(kx+1).$$

The final step is to prove that $k_0e = ke$, for, if this is the case, then indeed E equals $E(e,x,k)$. Now from (5.2), we have

$$0 = eq = e(k_0e-ky) = k_0e-k(ey) = k_0e-ke,$$

so $k_0e = ke$, as desired. $\square$

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