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Rings with internal cancellation

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Abstract

In this paper, we study the class of rings that satisfy internal direct sum cancellation with respect to their 1-sided ideals. These are known to be precisely the rings in which regular elements are unit-regular. Further characterizations for such "IC rings" are given, in terms of suitable versions of stable range conditions, and unique generator properties of idempotent generated right ideals. This approach leads to a uniform treatment of many of the known characterizations for an exchange ring to have stable range 1. Rings whose matrix rings are IC turn out to be precisely those rings whose finitely generated projective modules satisfy cancellation. We also offer a couple of "hidden" characterizations of unit-regular elements in rings that shed some new light on the relation between similarity and pseudo-similarity—in monoids as well as in rings. The paper concludes with a treatment of ideals for which idempotents lift modulo all 1-sided subideals. An appendix by R.G. Swan[1] on the failure of cancellation for finitely generated projective modules over complex group algebras shows that such algebras are in general not IC.
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1. Introduction

A module M_k over a ring k is said to satisfy *internal cancellation* (or we say M is *internally cancellable*) if, whenever $M = K \oplus N = K' \oplus N'$ (in the category of k-modules), $N \cong N' \Rightarrow K \cong K'$. An obvious necessary condition for this is that M_k be *Dedekind-finite*; that is, $M \cong M \oplus X \Rightarrow X = 0$. In general, however, this condition is only necessary, but not sufficient.

The starting point for this paper is the observation in [32] that "internal cancellation" is an "ER-property;" that is, a module-theoretic property that depends *only* on the endomorphism ring of the module. (For a more precise definition of an ER-property, see [32, (8.1)].) A proof for this using the notion of isomorphism between idempotents in rings was given in [32, (8.5)]. As was pointed out by the referee, this proof is just a special case of the equivalence of $\mathrm{add}(M)$ (the category of direct summands of finite direct sums of M) with the category of finitely generated projective modules over the endomorphism ring $\mathrm{End}(M)$, which was first observed by Dress [12]. However, yet another approach is possible, by finding directly a *purely ring-theoretic* condition on $\mathrm{End}(M)$ that characterizes the internal cancellability of M. This was essentially done by G. Ehrlich in [14]. To state Ehrlich's result, let us first recall some basic terminology for rings. An element a in any ring R is said to be *regular* (respectively *unit-regular*) if $a = axa$ for some $x \in R$ (respectively for some $x \in \mathrm{U}(R)$). (Throughout this paper, $\mathrm{U}(R)$ denotes the group of units of a ring R.) We shall write $\mathrm{reg}(R)$ (respectively $\mathrm{ureg}(R)$) for the set of regular (respectively unit-regular) elements of R. (It is easy to show that $a \in \mathrm{ureg}(R)$ iff a is the product of a unit and an idempotent, in either order; see [30, (4.14)(B)]. We shall use this fact freely below.) Finally, we say that R is (von Neumann) regular (respectively unit-regular) if $R = \mathrm{reg}(R)$ (respectively $R = \mathrm{ureg}(R)$). With these definitions, we have:

Ehrlich's Theorem 1.1. *Let $R = \mathrm{End}(M)$, where M is a right module over some ring k. Then M is internally cancellable iff* $\mathrm{reg}(R) = \mathrm{ureg}(R)$.

In [14], Ehrlich worked in the situation where the endomorphism ring R is regular. However, this assumption is not really necessary, and it was fairly well known that the essence of her arguments actually yielded the theorem above. An explicit statement of 1.1 appeared, for instance, in [20, p. 113]. In this paper, Guralnick and Lanski also found an important link between internal cancellation and the notion of *pseudo-similarity* in linear algebra. In any ring R, an element a is said to be pseudo-similar to an element b if there exist $x, y, z, w \in R$ such that

$$a = zbx, \qquad b = xaw, \quad \text{and} \quad x = xzx = xwx. \tag{1.2}$$

Clearly, if a, b are similar in R (that is, $a = u^{-1}bu$ for some $u \in \mathrm{U}(R)$), then a is pseudo-similar to b since (1.2) is satisfied by taking $x = u$ and $w = z = u^{-1}$. The converse does not hold in general. The main result in [20] is the following; see also [22, Theorem 1] and [24, Theorem 2B].

Theorem 1.3. *In the notation of Theorem* 1.1, *M is internally cancellable iff pseudo-similarity implies similarity in the ring $R = \mathrm{End}(M)$.*

The considerations above lead us directly to the class of rings R for which the right module R_R satisfies internal cancellation. We propose to call such rings "IC rings." *These are the rings R that arise as endomorphism rings of internally cancellable modules*. By what we have said so far, we have the following three equivalent criteria for a ring R to be IC:

(1.4) Isomorphic idempotents in R have isomorphic complementary idempotents; or equivalently, isomorphic idempotents are similar [30, Exercise 21.16].
(1.5) $\mathrm{reg}(R) = \mathrm{ureg}(R)$.
(1.6) Pseudo-similarity implies similarity in R.

Because of the characterization (1.5), IC rings have also been called *partially unit-regular rings* in [22]. However, we think the name "IC ring" is simpler and better. Note that any of the characterizations for IC rings above suffices to show that IC *is a left-right symmetric condition*. Needless to say, among von Neumann regular rings, the IC rings are just the unit-regular rings.

A ring R is said to be of stable range 1 (written $\mathrm{sr}(R) = 1$) if, for any $a, b \in R$, $aR + bR = R$ implies that $a + bx \in \mathrm{U}(R)$ for some $x \in R$. Modifying this definition, we say that R has *regular stable range* 1 (written $\mathrm{rsr}(R) = 1$) if the above implication holds for all $a, b \in \mathrm{reg}(R)$, or equivalently (as it turns out) for all $a \in \mathrm{reg}(R)$ and all $b \in R$. In Section 4 of this paper, we show that, in fact, *the condition* $\mathrm{rsr}(R) = 1$ *is a characterization for* IC *rings*. This result generalizes the well-known fact that a regular ring R is unit-regular iff $\mathrm{sr}(R) = 1$. Many variants for the condition $\mathrm{rsr}(R) = 1$ are possible. For instance, in 4.5, it is shown that $\mathrm{rsr}(R) = 1$ *iff, whenever $ar + e = 1$ where $a, r \in R$ and $e \in R$ is an idempotent, then $a + ex \in \mathrm{U}(R)$ for some $x \in R$.*

In Section 5, we study rings R that are *stably* IC, in the sense that $\mathbb{M}_n(R)$ is IC for all n. These turn out to be precisely the rings R whose category of finitely generated projective (say, right) modules satisfies cancellation. Examples of such rings include unit-regular rings, right free ideal rings, finite von Neumann algebras, and polynomial rings over commutative 0-dimensional rings and Dedekind rings, etc. A somewhat surprising result here is 5.10, which shows that a polynomial ring over even a stably IC ring need not be IC. The fact that every complex group algebra $\mathbb{C}G$ embeds into a finite "group von Neumann algebra" seems to suggest that $\mathbb{C}G$ might be (stably) IC. However, this turned out to be *not* the case, according to examples found by R.G. Swan, which (with his kind permission) are included in an Appendix to this paper.

In Section 6, we consider the problem of characterizing the generators of a principal right ideal aR in a ring R. We say that $a \in R$ has the right UG ("unique generator") property if, for any $b \in R$, $aR = bR$ implies that $b = au$ for some $u \in \mathrm{U}(R)$. In 6.2, we prove that R is IC iff all idempotents [or equivalently, all regular (respectively unit-regular) elements] have the right UG property. Using the various characterizations for IC rings above, we obtain easily many of the characterizations for an exchange ring R to have stable range 1. This is done in 6.5.

In Section 7, as a prelude to Section 8, we give a characterization for unit-regular elements in rings that seemed to have been narrowly missed in the literature. According to Theorem 7.1, *a regular element* $x = xyx$ (*in an arbitrary ring* R) *is unit-regular iff there exists a unit* $u \in R$ *such that* $xy = xu$ *and* $yx = ux$. This turns out to be a powerful tool in the study of pseudo-similarity. In Section 8, we compare the notion of pseudo-similarity (in four different flavors) with that of similarity, and use Theorem 7.1 to give a new conceptual treatment for the fact that IC rings are precisely those rings for which pseudo-similarity boils down to similarity. A couple of possibly surprising consequences of this study on the similarity of ring elements are given in 8.7 and 8.9. The paper concludes with Section 9, which offers some ostensibly weaker conditions characterizing IC rings through the use of the notion of "lifting ideals" (Theorem 9.7). The result 9.3 on lifting regular elements modulo a lifting ideal should be of independent interest.

The second author's expository article [32] provides much of the background for this work. In particular, for any undefined terms used in this paper, the reader should consult [32] (along with the ring theory texts [29] and [31]).

2. Examples of IC rings

The class of IC rings is quite broad. To show this, we assemble here an initial list of examples (and some non-examples) of IC rings. Example (6) below, however, is given without proof, and the justifications for example (7) will come later in Section 5.

Examples 2.1.

(1) We have observed at the beginning of the Introduction that an IC ring is always Dedekind-finite. However, a Dedekind-finite ring R need not be IC, even if R is von Neumann regular. For such an example, see [19, (5.10)].

(2) Since there do exist non IC rings, it is not surprising that we can write down a "generic" example. Let k be a field, and let R be the k-algebra generated by two elements a, b with the relations $(ab)^2 = ab$, and $(ba)^2 = ba$. In R, $e := ab$ and $e' := ba$ are isomorphic idempotents. *We claim that* R *is not an* IC *ring*. To see this, let S be the k-algebra generated by two elements x, y with the single relation $xy = 1$. The rules $a \mapsto x$, $b \mapsto y$ give a well-defined k-algebra homomorphism from R to S, since

$$(xy)^2 = 1 = xy \quad \text{and} \quad (yx)^2 = yxyx = yx.$$

Under this homomorphism, we have $1 - e \mapsto 1 - xy = 0$, and $1 - e' \mapsto 1 - yx \neq 0$. Since these two images are not isomorphic idempotents in S, it follows that $1 - e$ and $1 - e'$ are also non-isomorphic idempotents in R. Therefore, R is not IC. (It would be of interest to know if the ring R is Dedekind-finite. We do not know the answer.)

(3) As we have noted already, the criterion 1.5 implies that any unit-regular ring is IC.

(4) If R is an *abelian* ring (in the sense that all idempotents of R are central), isomorphic idempotents in R must be equal, according to [30, Exercise 22.2]. In particular, R is an IC ring. Examples of such rings include: all commutative rings, all reduced rings (e.g.,

strongly regular rings), and of course, all rings with only trivial idempotents $\{0, 1\}$. The latter class includes, for instance, all domains, local rings, and all group rings kG where k is any subring of $\mathbb{C}$ in which no prime number p is invertible (see [4, (8.11)]).

(5) *Any right artinian ring R is* IC. Indeed, if $e, e' \in R$ are idempotents in R such that $eR \cong e'R$, the classical Krull–Schmidt Theorem (applied to R_R) implies that $(1-e)R \cong (1-e')R$.

(6) (For the terminology used in this example, see [37].) *Any quasi-continuous ring is* IC. In fact, if R is a quasi-continuous ring, then R is also a CS ring [31, §6, Exercise 36], and therefore Dedekind-finite [31, Theorem 6.48]. According to a theorem of Müller and Rizvi (see, e.g., [37, (2.33)]), the Dedekind-finite quasi-continuous module R_R satisfies internal cancellation. Thus, R is an IC ring.

(7) Another major class of IC rings comes from the theory of operator algebras. By 5.12, *any finite von Neumann algebra is* IC.

Some of these examples will be further expanded and refined into a list of *stably* IC rings in 5.9.

3. Elements of stable range 1

We begin this section by taking the definition of stable range 1 and modifying it into an "element-wise" definition.

Definition 3.1. We say that an element a in a ring R has stable range 1 (written $\mathrm{sr}(a) = 1$, or, if necessary, $\mathrm{sr}_R(a) = 1$) if, for any $b \in R$, $aR + bR = R$ implies that $a + bx \in \mathrm{U}(R)$ for some $x \in R$.

Of course, with this definition, $\mathrm{sr}(R) = 1$ amounts to $\mathrm{sr}(a) = 1$ for all $a \in R$. The main result in this section gives (for any ring R) a natural class of elements with stable range 1.

Theorem 3.2. *For any ring R, $a \in \mathrm{ureg}(R)$ implies that* $\mathrm{sr}(a) = 1$. *In particular, all idempotents in R have stable range* 1.

The proof of this is based on two fairly standard lemmas, whose proofs are included here for completeness. The first one is the following fact about principal right ideals generated by unit-regular elements in an arbitrary ring R; see [24, Theorem 2B(14)].

Lemma 3.3. *If $a, a' \in \mathrm{ureg}(R)$, then $aR = a'R$ iff $a' = au$ for some $u \in \mathrm{U}(R)$.*

Proof. Let $a = ev$ and $a' = e'v'$, where e, e' are idempotents, and $v, v' \in \mathrm{U}(R)$. Since $aR = evR = eR$, and $a'R = e'v'R = e'R$, we have $aR = a'R$ iff $eR = e'R$. Thus, it suffices to handle the case where $a = e$ and $a' = e'$. We need only check the "only if" part, so assume that $eR = e'R$. Then $ee' = e'$, and $e'e = e$. Since $ee'(1-e)$ is an element of square zero, we have

$$u := 1 + ee'(1-e) = 1 + e' - e \in \mathrm{U}(R).$$

Now $eu = e(1 + e' - e) = e + e' - e = e'$, as desired. □

The second fact needed to prove Theorem 3.2 is the following lemma from [38, (2.8)].

Nicholson's Lemma 3.4. *Let P be a projective right module over any ring R, and let A, B be submodules of P such that $A + B = P$. If A is a direct summand of P, then there exists a submodule $C \subseteq B$ such that $P = A \oplus C$.*

Proof. Here is a proof of the lemma that seems easier than that given in [38]. Since P/A is a projective R-module, the projection map from P to P/A induces a split short exact sequence

$$0 \to A \cap B \to B \xrightarrow{\pi} P/A \to 0.$$

Thus, we are done by taking C to be the image of a splitting of π. □

Using 3.3 and 3.4, we can now give the proof for 3.2. Consider any $a \in \mathrm{ureg}(R)$, and let b be any element of R such that $aR + bR = R$. By the first part of the proof of 3.3, aR is a direct summand of R_R, so by Nicholson's Lemma, there exists a right ideal $C \subseteq bR$ such that $R = aR \oplus C$. Write (uniquely) $1 = e_1 + f_1$ where $e_1 \in aR$ and $f_1 \in C$. Then e_1, f_1 are complementary idempotents, with $aR = e_1R$ and $C = f_1R$ (see the solution to [30, Exercise 1.7]). Thus, 3.3 implies that $a = e_1u_1$ for some $u_1 \in \mathrm{U}(R)$. Writing $f_1 = by$ for some $y \in R$, and right-multiplying $1 = e_1 + f_1$ by u_1, we get $a + byu_1 = u_1 \in \mathrm{U}(R)$. This checks that $\mathrm{sr}(a) = 1$. □

After the writing of this paper, we found out that a stronger "1-sided version" of Theorem 3.2 has recently been proved by Li, Zhu, and Tong; see [35, Lemma 2]. The converse of 3.2 is easily seen to be false. However, a partial converse is true, as the following result shows.

Theorem 3.5. *Let $a \in \mathrm{reg}(R)$, where R is any ring. Then* $\mathrm{sr}(a) = 1$ *iff* $a \in \mathrm{ureg}(R)$.

Proof. The "if" part is true for any element $a \in R$, by 3.2. For the "only if" part, write $a = axa$ for some $x \in R$, and assume that $\mathrm{sr}(a) = 1$. The following familiar argument is from the proof of [19, (4.12)]. In view of $aR + (1 - ax)R = R$, we get an element $y \in R$ such that $a + (1 - ax)y \in \mathrm{U}(R)$. Letting u be the inverse of this unit, we have

$$a = axa = ax\big[a + (1 - ax)y\big]ua = axaua = aua,$$

so we have $a \in \mathrm{ureg}(R)$. □

For regular rings R, 3.5 recaptures the fact that R has stable range 1 iff it is unit-regular [32, (5.5)].

For later use in Section 6, we record in the following a necessary condition on elements of stable range 1 in any ring R.

Proposition 3.6. *Suppose* $\mathrm{sr}(a) = 1$ *for an element* $a \in R$. *Then, for any* $b \in R$, $bR = baR$ *iff* $ba = bu$ *for some* $u \in \mathrm{U}(R)$. *In particular, for any* $n \geqslant 1$, $a^n \in a^{n+1}R$ *iff* $a^n \in a^{n+1} \cdot \mathrm{U}(R)$.

Proof. We need only prove the "only if" parts. Assuming that $bR = baR$, we have $b = bar$ for some $r \in R$. Since $aR + (1 - ar)R = R$, the assumption that $\mathrm{sr}(a) = 1$ implies that $u := a + (1 - ar)s \in \mathrm{U}(R)$ for some $s \in R$. Left multiplying this equation by b, we have $bu = ba + (b - bar)s = ba$, as desired. The last conclusion follows by applying the first conclusion to the case where $b = a^n$. □

4. Rings with regular stable range 1

With the "element-wise" definition of stable range 1 (as in 3.1), it is natural to consider the following "regular version" of the condition $\mathrm{sr}(R) = 1$.

Definition 4.1. We say that a ring R has *regular stable range* 1 (written $\mathrm{rsr}(R) = 1$) if every $a \in \mathrm{reg}(R)$ has stable range 1. (Of course, $\mathrm{sr}(R) = 1 \Rightarrow \mathrm{rsr}(R) = 1$.)

We start by recording the following characterization of IC rings in terms of *regular* stable range 1, which we obtained in 2003. As it turned out, the same result has also been proved in [41].

Theorem 4.2. *A ring* R *is IC iff* $\mathrm{rsr}(R) = 1$.

Proof. First assume that R is IC, and consider any element $a \in \mathrm{reg}(R)$. Since $\mathrm{reg}(R) = \mathrm{ureg}(R)$, 3.2 gives $\mathrm{sr}(a) = 1$. This checks that $\mathrm{rsr}(R) = 1$. Conversely, assume that $\mathrm{rsr}(R) = 1$, and let $a \in \mathrm{reg}(R)$. By assumption, $\mathrm{sr}(a) = 1$, so by 3.5, $a \in \mathrm{ureg}(R)$. Thus, $\mathrm{reg}(R) = \mathrm{ureg}(R)$, and so R is an IC ring. □

In view of this theorem, it is of interest to give other descriptions for rings of regular stable range 1. Indeed, there is a rather large number of alternative descriptions! Consider the following general statement:

$$aR + bR = R \quad \Rightarrow \quad a + bx \in \mathrm{U}(R) \quad \text{for some } x \in R, \tag{$*$}$$

where the elements $a, b \in R$ are to be quantified. For each of a, b, we can use the quantifier "for all elements", or "for all regular elements", or "for all idempotents", so there are nine combinations. The combination "for all a, b" gives simply the condition $\mathrm{sr}(R) = 1$, and by 3.2, the three combinations arising from "for all idempotents a" lead to conditions that always hold. Excluding these four possibilities, we have five combinations left, which we spell out and explicitly label as follows:

(0) $(*)$ holds for all $a \in \mathrm{reg}(R)$ and $b \in R$. (This is the *definition* of $\mathrm{rsr}(R) = 1$.)
(1) $(*)$ holds for all $a, b \in \mathrm{reg}(R)$.

(2) $(*)$ holds for all $a \in \mathrm{reg}(R)$ and all idempotents $b \in R$.
(3) $(*)$ holds for all $a \in R$ and all idempotents $b \in R$.
(4) $(*)$ holds for all $a \in R$ and $b \in \mathrm{reg}(R)$.

Theorem 4.3. *The condition* $\mathrm{rsr}(R) = 1$ *is equivalent to each of* (0)–(4). *Alternatively, one can change* $(*)$ *into* $(**)$ *by replacing* $aR + bR = R$ *by an equation* $ar + b = 1$ (*where* $r \in R$), *and define the conditions* $(0)'$–$(4)'$ *accordingly, by using* $(**)$ (*instead of* $(*)$). *These five new conditions are also equivalent to* (0)–(4).

Proof. We first prove the equivalence of $(2)'$ with the conditions (0)–(4), by showing:

$$(0) \Rightarrow (1) \Rightarrow (2) \Rightarrow (2)' \Rightarrow (3) \Rightarrow (4) \Rightarrow (0).$$

Here, the only nontrivial implications are the last three, which we shall now verify.

$(2)' \Rightarrow (3)$. To check (3), start with $aR + eR = R$ where $e = e^2 \in R$. We have an equation $ar + es = 1$ for some $r, s \in R$. Let $a' := (1-e)a$. Left-multiplying $ar + es = 1$ by $1 - e$, we have $a'r = 1 - e$, and hence

$$a'ra' = (1-e)a' = (1-e)(1-e)a = a'.$$

Thus, $a' \in \mathrm{reg}(R)$. Since $a'r + e = 1$, $(2)'$ implies the existence of an element $y \in R$ such that

$$a' + ey = (1-e)a + ey = a + e(y - a)$$

is a unit, as desired.

$(3) \Rightarrow (4)$. This implication holds since, for any regular element $b = bcb \in R$, $e := bc$ is an idempotent, with $eR = bR$.

$(4) \Rightarrow (0)$. To check (0), start with $aR + bR = R$ where a is regular. As in the last implication, aR is generated by an idempotent, so it is a direct summand of R_R. By Nicholson's Lemma 3.4, there exists a right ideal $C \subseteq bR$ such that $R = aR \oplus C$. This enables us to write C as eR, for some idempotent $e \in R$. Since $e \in \mathrm{reg}(R)$, (4) implies that $a + ey \in \mathrm{U}(R)$ for some $y \in R$, and we are done by noting that $e \in bR$.

It is now easy to get the equivalence of $(0)'$–$(4)'$ with (0)–(4). Of course, we always have $(n) \Rightarrow (n)'$, for $0 \leqslant n \leqslant 4$. Since we have already dealt with $(2)'$, the rest follows by noting the trivial implications $(0)' \Rightarrow (1)' \Rightarrow (2)'$, and $(4)' \Rightarrow (3)' \Rightarrow (2)'$. □

Remark 4.4. Note that the proof of $(4) \Rightarrow (0)$ above used (4) only in the situation $aR \oplus eR = R$ (instead of $aR + eR = R$). This suggests that we can formulate five *more* conditions that are parallel to the conditions (0)–(4). This is done by replacing $aR + bR = R$ in $(*)$ by a direct sum equation $aR \oplus bR = R$. This leads to five new conditions $(0)''$–$(4)''$, which can be easily shown to be also equivalent to those in 4.3. The details of this verification are left to the reader.

Just for the record, we restate 4.2 and the equivalence (0) $\Leftrightarrow$ (3) $\Leftrightarrow$ (3)$'$ into the following explicit characterization of IC rings. This characterization has been obtained earlier by H. Chen [10, Lemma 1].

Corollary 4.5. *R is an IC ring iff, for any idempotent $e \in R$, $aR + eR = R$ (or alternatively, $ar + e = 1$) implies that $a + ex \in \mathrm{U}(R)$ for some $x \in R$.*

This corollary was first proved *for exchange rings R* by H.-P. Yu (see (4) $\Leftrightarrow$ (2) in [46, Theorem 9]). The approach in [10] and in this paper gives the result for *all* rings R.

5. Functorial behavior of IC, and stably IC rings

In the first half of this section, we shall say a few things about how IC rings behave with respect to some of the standard constructions in ring theory, such as direct products, limits, subrings and factor rings, corner rings, polynomial rings and matrix rings, etc. The second half of the section will be devoted to the study of the notion of *stably* IC rings.

To begin with, IC rings are clearly closed under the formation of direct products and direct limits, in view of, say, the criterion (1.5). For subrings, we have the following partial result.

Proposition 5.1. *Let S be a (unital) subring in an IC ring R. If $R = S \oplus J$ for some ideal $J \subseteq R$, then S is also IC.*

Proof. Let e, e' be a pair of isomorphic idempotents in S. Then, e, e' are also isomorphic in R, and so $1 - e, 1 - e'$ are isomorphic in R (by 1.4). Applying the natural ring homomorphism from R to $R/J \cong S$, we see that $1 - e, 1 - e'$ are also isomorphic in S. This checks that S is an IC ring. $\square$

Without assuming the existence of the ideal J, 5.1 need not hold. In fact, in Goodearl's book [19, (5.12)], there is an example of a unit-regular ring R which has a regular unital subring S that is not unit-regular. In this example, therefore, R is IC, but its unital subring S is not.

In spite of 5.1, a *general* factor ring of an IC ring need not be IC. This can be seen by taking the free algebra $R = \mathbb{Q}\langle x, y\rangle$. As a domain, R is an IC ring, but its factor ring obtained by using the relation $xy = 1$ is not Dedekind-finite, and so cannot be an IC ring. Under suitable assumptions, however, we can get some positive results, as in (2) below.

Proposition 5.2. *Let J be an ideal in a ring R, and let $\overline{R} := R/J$.*

(1) *If $J \subseteq \mathrm{rad}(R)$ and $\overline{R}$ is IC, then R is IC.*
(2) *Assume that either $J \subseteq \mathrm{reg}(R)$, or $J \subseteq \mathrm{rad}(R)$ and idempotents of $\overline{R}$ can be lifted to R. If R is IC, then so is $\overline{R}$.*

Proof. (1) Suppose e, e' are isomorphic idempotents in R. Then $\bar{e}$ and $\overline{e'}$ are isomorphic in $\overline{R}$, and so (assuming that $\overline{R}$ is IC) $1-\bar{e}, 1-\overline{e'}$ are isomorphic in $\overline{R}$. Since $J \subseteq \mathrm{rad}(R)$, [29, (21.21)] implies that $1-e$ and $1-e'$ are isomorphic in R. This checks that R is IC.

(2) Now assume R is IC. If $J \subseteq \mathrm{rad}(R)$ and idempotents in $\overline{R}$ can be lifted to R, the same argument as in (1) shows that $\overline{R}$ is IC. Next, assume that $J \subseteq \mathrm{reg}(R)$. To see that $\overline{R}$ is IC, it suffices to check the equation $\mathrm{reg}(\overline{R}) = \mathrm{ureg}(\overline{R})$. Let $a \in R$ be such that $\bar{a} \in \mathrm{reg}(\overline{R})$, say $\bar{a} = \bar{a}\bar{x}\bar{a}$, for some $x \in R$. Then $a - axa \in J \subseteq \mathrm{reg}(R)$, so there exists $y \in R$ such that

$$a - axa = (a - axa)y(a - axa) = a(1 - xa)y(1 - ax)a \in aRa.$$

This gives $a \in aRa$, so $a \in \mathrm{reg}(R)$. Since R is IC, we have (by (1.5)) $a = aua$ for some $u \in \mathrm{U}(R)$. Then $\bar{a} = \bar{a}\bar{u}\bar{a}$ with $\bar{u} \in \mathrm{U}(\overline{R})$, so we have $\bar{a} \in \mathrm{ureg}(\overline{R})$, as desired. □

We will come back in Section 9 to give some more results relating the IC properties of R and R/J in the case where J may not be contained in $\mathrm{rad}(R)$; see 9.7. At this point, let us record some easy consequences of 5.2. We suppress the details on the first corollary below since this has already been proved (via different methods) by H. Chen [10, Corollaries 8, 10].

Corollary 5.3.

(1) *Let* $R = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$, *where* A, B *are rings, and* M *is an* (A, B)*-bimodule. Then* R *is IC iff* A, B *are both IC.*
(2) *A ring* R *is IC iff the ring of* $n \times n$ *upper triangular matrices over* R *is IC* (*for any fixed* n).

Corollary 5.4.

(1) *A ring* S *is IC iff the power series ring* $R = S[\![x]\!]$ *is IC.*
(2) S *is IC if the polynomial ring* $S[x]$ *is IC.*

Proof. The two "if" parts follow from 5.1. The "only if" part in (1) follows from 5.2(1) since the ideal $J \subseteq R$ generated by x lies in $\mathrm{rad}(R)$ [30, Exercise 5.6], and $R/J \cong S$. □

Remark 5.5. It may be tempting to think that (2) above could be turned into an "if and only if" statement. Unfortunately, this is not the case, as we shall show in 5.10 below. (On the other hand, some positive cases are indicated in 5.9(7).)

As for the formation of Peirce corner rings, we have the following easy result.

Proposition 5.6. *If* R *is an IC ring, then so is any Peirce corner ring* eRe (*for any idempotent* $e \in R$).

Proof. By definition, R being IC means that the module R_R is internally cancellable. Since $R_R = eR \oplus (1-e)R$, we see easily that its direct summand $(eR)_R$ is also internally can-

cellable. Since "internal cancellation" is an ER-property, it follows that the endomorphism ring $\mathrm{End}_R(eR) \cong eRe$ is an IC ring. □

Of course, 5.6 can also be stated in the form that $\mathrm{reg}(R) = \mathrm{ureg}(R)$ implies $\mathrm{reg}(eRe) = \mathrm{ureg}(eRe)$. In the case of regular rings, this recaptures the familiar fact that *a Peirce corner ring of a unit-regular ring is also unit-regular* [32, (5.5)(6)].

It is well known that "stable range 1" is a Morita-invariant property of rings; for a proof of this, see [32, (5.6)]. Proposition 5.6 would seem to be the first step toward proving a similar result for "regular stable range 1" (or equivalently, the IC property for rings). However, the next proposition (and the subsequent examples) will show that this is not the case. Here and in the following, we write $\mathcal{P}(R)$ for the category of finitely generated projective right modules over R.

Proposition 5.7. *For any ring R, if $\mathbb{M}_n(R)$ is* IC*, then so is $\mathbb{M}_k(R)$ for any $k \leqslant n$. In general, the following statements are equivalent*:

(1) *R is stably* IC*; that is, $\mathbb{M}_n(R)$ is* IC *for all n.*
(2) $\mathrm{End}_R(P)$ *is* IC *for any P in $\mathcal{P}(R)$.*
(3) *The category $\mathcal{P}(R)$ satisfies cancellation.*
(4) *The module R_R is cancellable in the category $\mathcal{P}(R)$.*

Any ring R satisfying any of these conditions is stably finite; *that is, $\mathbb{M}_n(R)$ is Dedekind-finite for all $n \geqslant 1$.*

Proof. Since $\mathbb{M}_n(R) \cong \mathrm{End}(R_R^n)$, the condition that $\mathbb{M}_n(R)$ be IC means that the module R_R^n satisfies internal cancellation. This implies that R_R^k for any $k \leqslant n$ also satisfies internal cancellation, so the first conclusion in the proposition follows.

For any R, we clearly have (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4), and (1) $\Leftrightarrow$ (2) follows from the fact that any direct summand of an internally cancellable module remains internally cancellable. Thus, it only remains to prove that (4) $\Rightarrow$ (3). Now if R_R is cancellable in $\mathcal{P}(R)$, then so is R_R^n (for any natural number n) and every finitely generated projective right R-module. The latter clearly implies (3).

The last conclusion of the proposition follows from the fact that IC rings are Dedekind-finite. □

Most parts of Theorem 5.7 have also been observed in the paper of Song, Chu, and Zhu [41]. Taken as a whole, this theorem suggests the utility of defining an "IC-level" of a ring R. If R is not IC, we define its IC-level to be 0. Otherwise, we define the IC-level of R to be

$$\sup\bigl\{n\colon \mathbb{M}_n(R) \text{ is IC}\bigr\} \geqslant 1.$$

In this terminology, rings of infinite IC-level are just the stably IC rings, or equivalently, rings satisfying (2)–(4) in Proposition 5.7. If, instead, a ring R has a finite IC-level n, then the matrix rings $\mathbb{M}_k(R)$ are IC for all $k \leqslant n$, and not IC for all $k > n$.

In the case of a *commutative* ring R, there are some special results about cancellation in $\mathcal{P}(R)$ that are useful toward the computation of the IC-level of R. The first of these is the fact that, for A, B, C in $\mathcal{P}(R)$ with B, C of rank 1,

$$A \oplus B \cong A \oplus C \quad \Rightarrow \quad B \cong C. \tag{5.8}$$

To prove this, we may assume that $A \cong R^{n-1}$ (for some $n \geqslant 2$). After this, (5.8) follows by taking the nth exterior powers (as in the proof of [33, (I.4.11)]). The second nice fact is Bass's Cancellation Theorem [3, Theorem (9.3)], which states that (5.8) always holds if R is a commutative noetherian ring of dimension d, and A, B, C are in $\mathcal{P}(R)$ with B, C of rank $> d$. These facts quickly lead to (1) and (2) of the following examples on the IC-level.

Examples 5.9.

(1) Over any commutative ring R with only trivial idempotents, any (projective) module in $\mathcal{P}(R)$ has a constant rank. In view of this, it follows easily from 5.8 that *any rank* 2 *module* P *in* $\mathcal{P}(R)$ *is internally cancellable*. Taking P to be R^2, we see, in particular, that R has IC-level $\geqslant 2$.

(2) If R is a commutative 1-dimensional noetherian ring with only trivial idempotents, the two cancellation results recalled before 5.9 imply that $\mathcal{P}(R)$ satisfies cancellation, and thus by 5.7, R has IC-level ∞ (that is, R is stably IC). However, if R is the (2-dimensional) coordinate ring of the real 2-sphere, then according to [42], R_R^3 does not satisfy internal cancellation. Therefore, R has IC-level exactly equal to 2. (From this, it also follows that $\mathbb{M}_2(R)$ has IC-level 1.)

(2)′ Let R be the Witt ring $W(F)$ of quadratic forms over a field F of characteristic $\neq 2$. Since unary forms generate $W(F)$ and they have square 1, $W(F)$ has Krull dimension $\leqslant 1$. But $W(F)$ may not be noetherian, so Bass's Cancellation Theorem cannot be applied directly. Nevertheless, in [18], Fitzgerald has studied the structure of f.g. projective modules over $W(F)$, and has essentially proved that the category of such modules satisfy cancellation. Thus, $W(F)$ is a stably IC ring. (The fact that $W(F)$ is an IC ring is much easier, as it is well known that $W(F)$ has no nontrivial idempotents: see [34, VIII.8.6].)

(3) If G is the generalized quaternion group of order 32, then the integral group ring $R := \mathbb{Z}[G]$ is IC (see 2.1(4), or [29, (8.26)]), but $(R^2)_R$ is not internally cancellable according to Swan [43, Theorem 3]. Thus, R again has IC-level 1. The examples in (2) and (3) show, in particular, that being IC is *not* a Morita-invariant property of rings. On the other hand, it is easy to show, using 5.6, that being *stably* IC is a Morita-invariant property.

(4) All rings R with $\mathrm{sr}(R) = 1$ are stably IC (i.e., have infinite IC-level), since R satisfies (4) in 5.7 in a stronger form: by [32, (5.4)], R_R is cancellable in the category of *all* right R-modules. In particular, *semilocal rings, unit-regular rings, self-injective rings, and strongly π-regular rings are all stably* IC: see, respectively [29, (20.9)], [32, (5.5)], [32, (7.17)], and [1]. (The "self-injective" case can also be deduced from 2.1(6), since self-injective rings are quasi-continuous, and self-injectivity goes up to matrix rings.)

(5) Let R be a module-finite algebra over a commutative ring of Krull dimension 0. By [30, (23.10)], R is strongly π-regular, and hence stably IC by (4).
(6) From [32, (3.7)(B)], any right free ideal ring ("right FIR") is also stably IC.
(7) (extending Corollary 3 in [20]). Let R be any stably IC ring with the property that, for any m, any module in $\mathcal{P}(R[x_1,\dots,x_m])$ is extended (by tensoring) from an R-module (necessarily in $\mathcal{P}(R)$). Then $\mathcal{P}(R[x_1,\dots,x_m])$ satisfies cancellation for $m=0$, and hence for all m. Therefore, by 5.7, any polynomial ring $R[x_1,\dots,x_m]$ is stably IC. Since

$$\mathbb{M}_n\big(R[x_1,\dots,x_m]\big) \cong \mathbb{M}_n(R)[x_1,\dots,x_m],$$

it follows that, for any n, *any polynomial ring over* $\mathbb{M}_n(R)$ *is* IC. Note that there is no lack of examples of rings R satisfying the hypotheses of (7). For instance, R can be any commutative 0-dimensional ring, or any Dedekind domain, or any 2-dimensional commutative noetherian ring with stable range 1. (For these rings, the extendibility of modules of $\mathcal{P}(R[x_1,\dots,x_m])$ from R follows from the results of Quillen and Suslin on Serre's Conjecture; see [33].)

Of course, we do not expect the conclusions in (7) above to hold if it is not given that the modules in $\mathcal{P}(R[x_1,\dots,x_m])$ are extended from R. Capitalizing on this line of reasoning, we actually arrive at a (strongly) negative answer to the question whether polynomial rings over IC rings remain IC.

Proposition 5.10. *There exists a stably IC ring R such that the polynomial ring $R[y]$ is not IC.*

Proof. Let D be any noncommutative division ring (e.g., Hamilton's ring of real quaternions), and let $R=\mathbb{M}_n(D[x])$, where $n\geqslant 2$. Since $D[x]$ is a PRID (principal right ideal ring), all modules in $\mathcal{P}(D[x])$ are free. By 5.7, this implies that $D[x]$ is stably IC, so R is IC. Since

$$\mathbb{M}_r(R)=\mathbb{M}_r\big(\mathbb{M}_n\big(D[x]\big)\big)\cong\mathbb{M}_{rn}\big(D[x]\big),$$

it follows that R is in fact *stably* IC. But

$$R[y]=\mathbb{M}_n\big(D[x]\big)[y]\cong\mathbb{M}_n\big(D[x,y]\big), \tag{5.11}$$

and by a result of Ojanguren and Sridharan ([39], [33, (II.3)], [44, Lemma 3.1]), the polynomial ring $S:=D[x,y]$ has a non-principal right ideal P such that $P\oplus S\cong S\oplus S$ in $\mathcal{P}(S)$. This implies that S_S^2 is not internally cancellable. Since $n\geqslant 2$, the isomorphism in (5.11) implies that the ring $R[y]$ is not IC. (Of course, for $n=1$, $R[y]=D[x,y]$ would have been IC, since it is a domain.) □

To get another large class of stably IC rings, let us now turn our attention to the theory of operator algebras. Let $(R,{}^*)$ be a von Neumann algebra in the algebra $\mathcal{B}(\mathcal{H})$ of bounded

operators on a Hilbert space $\mathcal{H}$, where * means the adjoint. By a *projection* in R, we mean an idempotent $p \in R$ such that $p = p^*$. Two projections $p, p' \in R$ are said to be *equivalent* (written $p \approx p'$) if there exists $q \in R$ such that $p = qq^*$ and $p' = q^*q$. (In particular, p, p' are isomorphic as idempotents.) Finally, a von Neumann algebra R is said to be *finite* if the only projection $p \approx 1$ in R is equal to 1. (Here, of course, "1" means the identity operator on $\mathcal{H}$.) We thank Ioannis Emmanouil for pointing out to us the following result of Lück ([36, Corollary 3.2], combined with 5.7).

Theorem 5.12. *For any von Neumann algebra $(R,^*) \subseteq \mathcal{B}(\mathcal{H})$, the following are equivalent*:

(1) $(R,^*)$ *is a finite von Neumann algebra*;
(2) *the ring R is Dedekind-finite*;
(3) *the ring R is* IC;
(4) *the ring R is stably* IC.

In particular, R can only have IC-*level* 0 *or* ∞.

Proof. For the sake of completeness, we record a proof here along the lines suggested by Emmanouil. We have clearly (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1), so it is enough to prove (1) $\Rightarrow$ (4). Assuming that the von Neumann algebra $(R,^*)$ is finite, the main tool for proving that R is (stably) IC is the existence of a "center-valued trace" on R. Let $Z(R)$ denote the center of R. By [25, (8.4.3)], there exists a $Z(R)$-linear mapping $\Delta : R \to Z(R)$ such that

(1) Δ *is the identity on* $Z(R)$;
(2) $\Delta(ab) = \Delta(ba)$ *for all* $a, b \in R$; *and*
(3) *for any projections* $p, p' \in R : p \approx p' \Leftrightarrow \Delta(p) = \Delta(p')$.

To check that R is IC, we verify the condition (1.4) for R as follows. Let e, e' be isomorphic idempotents in $(R,^*)$. By [27, Theorem 26], there exist projections $p, p' \in R$ such that $eR = pR$ and $e'R = p'R$. Then p, p' are isomorphic as idempotents, and hence (by (2)) $\Delta(p) = \Delta(p')$. By (1) (and linearity), we have

$$\Delta(1-p) = 1 - \Delta(p) = 1 - \Delta\big(p'\big) = \Delta\big(1-p'\big),$$

and so by (3), $1 - p \approx 1 - p'$. It follows that

$$(1-e)R \cong R/eR = R/pR \cong (1-p)R \cong \big(1-p'\big)R \cong R/p'R = R/e'R \cong \big(1-e'\big)R.$$

This checks that R is IC.[2] Since $\mathbb{M}_n(R)$ is also a finite von Neumann algebra (on the Hilbert space $\mathcal{H}^n$: see, e.g., [15, (2.3)(1)]), the above implies that this matrix algebra is IC, and hence R is stably IC. $\square$

[2] In retrospect, the use of the trace Δ lies in its main consequence that (in a *finite* von Neumann algebra) two projections are equivalent iff they are isomorphic as idempotents

Theorem 5.12 should be a very useful result, since it implies that *the various properties obtained for* IC *rings in this paper, including those in the remaining sections, are applicable to all finite von Neumann algebras*. For instance, applied in conjunction with 1.3 (in the case $M = R_R$), Theorem 5.12 yields the following nice result by purely algebraic means:

Corollary 5.13. *In any finite von Neumann algebra $R \subseteq \mathcal{B}(\mathcal{H})$, regular operators* (*in the sense of von Neumann*) *are unit-regular, and pseudo-similarity of operators* (*in the sense of* (1.2)) *is equivalent to similarity of operators.*

Our interest in finite von Neumann algebras stems in part from complex group algebras. For any (multiplicative) group G, the complex group algebra $\mathbb{C}G$ acts by left translation on the Hilbert space $\ell^2 G$, leading to an embedding $\mathbb{C}G \hookrightarrow \mathcal{B}(\ell^2 G)$. The closure of $\mathbb{C}G$ with respect to the weak operator topology, denoted by $\mathcal{N}G$, is called the *group von Neumann algebra* of G. It is well known that $\mathcal{N}G$ is always a *finite* von Neumann algebra. Therefore, by 5.12, $\mathbb{C}G$ is embedded in the stably IC ring $\mathcal{N}G$. This implies, for instance, that *any matrix ring $\mathbb{M}_n(\mathbb{C}G)$ is Dedekind-finite*, which is a famous result of Kaplansky. However, while the Dedekind-finite property descends to subrings, the IC property in general does not. Thus, it is not clear (from the above embedding) what can be said about the IC property of $\mathbb{C}G$.

Several classes of group algebras $A = \mathbb{C}G$ do turn out to be stably IC. For instance, if G is a locally finite group, we can reduce the considerations to the case where G is finite. In that case, all matrix rings over A are artinian, so A is stably IC by 2.1(5). If G is an abelian group, we may again assume that it is finitely generated, and an easy argument enables us to replace A by a Laurent polynomial ring B in finitely many commuting variables over a field. In this case, by an extension of Serre's Conjecture due to Swan (see [33, (V.4.10)]), all finitely generated projective B-modules are free, so by 5.7, B (and hence A) is stably IC. Finally, if G is a free group, it is known that A is a right FIR, and hence A is also stably IC by 5.9(6).

In view of Kaplansky's theorem and the examples in the above paragraph, we had speculated earlier that complex group algebras are (stably) IC. However, Professor Swan has found counterexamples to this statement. These counterexamples appear in the Appendix. Given such counterexamples, it would seem all the more interesting to ask for what groups G will the group algebra $\mathbb{C}G$ be IC or stably IC. For instance, a very challenging case is that of torsionfree groups G, for which it is already unknown if the group algebra is a domain (see, e.g., [31, pp. 90–98]).

In closing our discussion of group algebras, it is of interest to point out that, besides using the group von Neumann algebra $\mathcal{N}G$, there is a second way to embed $\mathbb{C}G$ into a stably IC ring, and this works for group algebras kG over any field k of characteristic zero. In fact, in this case, C. Faith [17] observed that the maximal left and right quotient rings of $A = kG$ are the same, and this common maximal quotient ring $Q_{\max}(A)$ is self-injective and von Neumann regular. By [19, (9.29)], $Q_{\max}(A)$ must then be unit-regular. Thus, by 5.9(4), $Q_{\max}(A)$ is a stably IC ring, in which the group algebra $A = kG$ embeds.

For other examples of non-cancellation of projective modules over algebras related to complex group algebras, see [40].

6. The unique generator property

This section is partly motivated by Lemma 3.3, which states that if a principal right ideal in any ring is generated both by a and a' in $\mathrm{ureg}(R)$, then a, a' are right associates of each other. In general, this statement does not hold if even one of a, a' fails to be unit-regular. This observation leads to the following definition, which has its origin in Kaplansky's classic paper [26].

Definition 6.1. An element $a \in R$ is said to have the *right unique generator property* (or "right UG" for short) if, for any $b \in R$, $aR = bR$ implies that $b = au$ for some $u \in \mathrm{U}(R)$. If all elements $a \in R$ have this property, we say that R has right UG. (For instance, all domains have (left and right) UG.)

Regarding this definition, we should point out that it is apparently unknown whether "right UG" is equivalent to "left UG;" see [6, Remark 4.9]. For some partial results in this direction, see the last parts of 6.2 and 6.5.

The element-wise right UG property in 6.1 leads to yet another group of characterizations for IC rings. These characterizations are to a large extent motivated by [24, Theorem 2B(14)]; in fact, (2) $\Rightarrow$ (1) below can be gleaned from this reference.

Theorem 6.2. *For any ring R, the following are equivalent*:

(1) *R is* IC.
(2) *Every regular element in R has right* UG.
(3) *Every unit-regular element in R has right* UG.
(4) *Every idempotent in R has right* UG.

In particular, the conditions (2)–(4) *are left-right symmetric.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are tautologies.

(4) $\Rightarrow$ (1). To check (1), we verify the condition $\mathrm{reg}(R) = \mathrm{ureg}(R)$. Given $x \in \mathrm{reg}(R)$, write $x = xyx$ (for some $y \in R$). Then xy is an idempotent, and $xR = xyR$. By (4), we have therefore $xy = xv$ for some $v \in \mathrm{U}(R)$, and hence $x = xyx = xvx \in \mathrm{ureg}(R)$.

(1) $\Rightarrow$ (2). Suppose $xR = zR$, where $x \in \mathrm{reg}(R)$. By (1.5), we can write $x = xyx$ for some $y \in \mathrm{U}(R)$. As in the above, $xR = eR$, where $e := xy$ is an idempotent. Since $ex = x$, $z \in xR$ implies that $ez = z$ also. Now

$$zR + (1-e)R = xR + (1-e)R = R,$$

and $1 - e$ is an idempotent. Thus, 4.5 yields a unit $u = z + (1-e)r$ for some $r \in R$. Left-multiplying this equation by e, we get $eu = ez = z$, and thus $z = x(yu)$ with $yu \in \mathrm{U}(R)$, as desired. □

Corollary 6.3. *For any ring R, we have*

$$\mathrm{sr}(R) = 1 \quad \Rightarrow \quad R \textit{ has right } \mathrm{UG} \quad \Rightarrow \quad R \textit{ is an } \mathrm{IC} \textit{ ring.}$$

However, neither implication is reversible.

Proof. The second implication follows tautologically from 6.2. The first implication is due to M. Canfell [6, (4.5)]. For the sake of completeness, we shall include a quick proof. Suppose $bR = b'R$, where $\mathrm{sr}(R) = 1$. Writing $b' = ba$ for some $a \in R$, we have $\mathrm{sr}(a) = 1$. Thus, 3.6 implies that $b' = ba = bu$ for some $u \in \mathrm{U}(R)$.

To check the last statement in the Corollary, note that the ring $\mathbb{Z}$ has UG, but does not have stable range 1. On the other hand, commutative rings are always IC, but they need not have (right) UG, according to Kaplansky's example [26, (b), p. 466]:

$$R := \{(n, f(x)) \in \mathbb{Z} \times \mathbb{Z}_5[x]\colon\ f(0) \equiv n \pmod 5\}, \tag{6.4}$$

in which $(0, x)$ and $(0, 2x)$ generate the same ideal but are not associates. (The ring R also has only trivial idempotents, which gives a second reason for it to be IC.) □

As a by-product of our discussions, we give a uniform treatment for some of the main criteria for exchange rings to have stable range 1 obtained in [5,46], and [7–9]. The main advantage of our approach is that, after we have developed the basic properties of IC rings, various criteria for exchange rings to have stable range 1 are now automatic consequences.

According to Warfield [45], a ring R is an *exchange ring* if the module R_R (or equivalently, the module ${}_RR$) has the exchange property. For a quick introduction to modules with the exchange or finite exchange properties, see [32, §6]. The following result unifies the criteria for an exchange ring to have stable range 1 obtained in [46, Theorem 9], [5, Theorem 3], [7, Proposition 3.5], [8, Lemma 1.1], and [9, Theorem 6].

Theorem 6.5. *For any exchange ring R, the following are equivalent*:

(A) $\mathrm{sr}(R) = 1$.
(B) *R has right* UG.
(C) *R is* IC.
(D) *R is stably* IC.
$(\mathrm{E})_1$ *R_R is cancellable in the category of right R-modules.*
$(\mathrm{E})_2$ *R_R is cancellable in the category $\mathcal{P}(R)$.*
$(\mathrm{E})_3$ *The category $\mathcal{P}(R)$ satisfies cancellation.*

In particular, an exchange ring R has left UG *iff it has right* UG, *and R can only have* IC-*level* 0 *or* ∞.

Proof. Without assuming R to be an exchange ring, we have (by 6.4) (A) ⇒ (B) ⇒ (C), and by [32, (4.2), (4.4)] and 5.7,

$$(\mathrm{A}) \quad\Rightarrow\quad (\mathrm{E})_1 \quad\Rightarrow\quad (\mathrm{E})_2 \quad\Rightarrow\quad (\mathrm{E})_3 \quad\Rightarrow\quad (\mathrm{D}) \quad\Rightarrow\quad (\mathrm{C}).$$

Therefore, the crucial step for proving the theorem is to show that, *for an exchange ring R*, we have (C) ⇒ (A). (This implication is, in particular, the *only* place in the proof where we'll use the exchange ring assumption!)

Assume (C), and let $a, b \in R$ be such that $aR + bR = R$. Since R is an exchange ring, there exists an idempotent of the form $e = by \in R$ such that $R = aR + eR$ (see [38, (2.9)]). Applying 4.5, we get an element $x \in R$ such that $a + ex = a + byx \in \mathrm{U}(R)$. This checks that $\mathrm{sr}(R) = 1$, proving that (C) $\Rightarrow$ (A). □

Remark 6.6.

(1) In retrospect, it is not surprising that (C) $\Leftrightarrow$ (D) in 6.5, since both the exchange ring condition and the stable range 1 condition are known to go up to matrix rings.
(2) In [38], Nicholson called a ring R *potent* if idempotents in $R/\mathrm{rad}(R)$ can be lifted to R and any right ideal of R not contained in $\mathrm{rad}(R)$ contains a nonzero idempotent. The class of potent rings contains the class of exchange rings. However, Theorem 6.5 does not hold for potent rings. In fact, Nicholson's example of a non-exchange potent ring, the subring $R \subseteq \mathbb{Q} \times \mathbb{Q} \times \cdots$ consisting of sequences of the form

$$(x_1, \ldots, x_n, s, s, s, \ldots) \quad (n \geqslant 1,\ x_i \in \mathbb{Q},\ s \in \mathbb{Z})$$

is commutative and hence IC, but has an epimorphic image $\cong \mathbb{Z}$, so it cannot have stable range 1.

We can now also deduce the following result of Yu ([46, Theorem 10], [32, (6.11)]) on finite exchange modules. The terminology and basic approach here follow that of [32].

Theorem 6.7. *Let M be a module with finite exchange* (*over any ring*). *Then M has the substitution property iff it is cancellable, iff it is internally cancellable.*

Proof. Since the substitution property implies the cancellation property, which in turn implies the internal cancellation property [32, (4.2)], it is sufficient to prove that the internal cancellation property on M implies the substitution property. Since both of these are ER-properties, it suffices to prove this for the exchange module R_R, where $R = \mathrm{End}(M)$. In this case, $\mathrm{sr}(R) = 1$ by (C) $\Rightarrow$ (A) in 6.5 and thus R_R has the substitution property by [32, Theorem 4.4]. □

7. "Hidden" characterizations of unit-regular elements

Unit-regular elements in rings have been studied a great deal in the literature, starting with Ehrlich's original work [13,14]. Many characterizations of unit-regular elements are known. Besides those given in [13] and [14], for instance, there is an extensive list of some twenty characterizations for unit-regular elements in rings given in Theorems 2A and 2B in the paper of Hartwig and Luh [24]. In the present work, we have made further additions to this list: for instance, in Section 3, we saw that unit-regular elements in a ring R are just elements in $\mathrm{reg}(R)$ that have stable range 1. In this section, we return to the work of Ehrlich, and extract from it a characterization of unit-regular elements that seems to have (so far) remained "hidden": *if $x = xyx \in R$, then $x \in \mathrm{ureg}(R)$ iff there exists a unit $u \in \mathrm{U}(R)$ such*

that $xy = xu$ *and* $yx = ux$. If we omit the second equation here, we would indeed get a valid characterization for x to be unit-regular, by an easy application of Lemma 3.3. However, if we insist on the pair of equations, $xy = xu$ and $yx = ux$ (for a *single* unit u), the "necessity" part is no longer obvious. This result, which we shall prove in 7.1 below, will be used in Section 8 to give a new treatment of the notion of pseudo-similarity in rings, culminating in a natural proof for the criterion 1.6: that IC rings are precisely those rings in which pseudo-similarity is equivalent to similarity.

For convenience, we shall state the intended characterization for unit-regular elements as one of several equivalent criteria. Note that the equivalence (1) $\Leftrightarrow$ (5) below is due to Hartwig and Luh, in [24, Theorem 2B]. We have included their characterization (5) here because of its obvious kinship with the other characterizations.

Theorem 7.1. *For a regular element* $x = xyx$ *in any ring* R*, the following conditions are equivalent*:

(1) $x \in \operatorname{ureg}(R)$.
(2) *There exists* $u \in \mathrm{U}(R)$ *such that* $xy = xu$.
(3) *There exists* $u \in \mathrm{U}(R)$ *such that* $xy = xu$ *and* $yx = ux$.
(4) *There exist* $u \in \mathrm{U}(R)$ *and* $r \in R$ *such that* $xy = ru$ *and* $yx = ur$.
(4)′ *There exist* $u' \in \mathrm{U}(R)$ *and* $r' \in R$ *such that* $xy = u'r'$ *and* $yx = r'u'$.
(5) xy *is similar to* yx *in* R.

Proof. To begin with, we note that the conditions (4), (4)′, and (5) are equivalent, even in the case where xy and yx are replaced by two arbitrary elements in R. Next, (3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are tautologies, and (2) $\Rightarrow$ (1) is obvious. Therefore, it suffices for us to prove (1) $\Rightarrow$ (3) and (5) $\Rightarrow$ (1), which are, of course, the more interesting parts of the theorem!

(5) $\Rightarrow$ (1). We can quote [24] here, but the proof of this implication in [24, Theorem 2B] involves a number of other equivalent conditions not stated here. For the sake of completeness, we shall offer a direct proof. (The main idea of this proof will also be used in the proof of (1) $\Rightarrow$ (3) below.) To verify (1), we represent R as the endomorphism ring of a right module M over some ring k (e.g., $M = R_R$, with elements of R acting by left multiplication). Since $x = xyx$, we have the following standard direct sum decompositions of k-modules (see the solution to [30, Exercise 4.14A$_1$]):

$$M = \ker(x) \oplus \operatorname{im}(yx) \quad \text{and} \quad M = \ker(xy) \oplus \operatorname{im}(x). \tag{$*$}$$

From this, $\operatorname{coker}(x) = M/\operatorname{im}(x) \cong \ker(xy)$. Since xy is an idempotent, the latter is isomorphic to $M/\operatorname{im}(xy)$. Now, by (5), $M/\operatorname{im}(xy) \cong M/\operatorname{im}(yx)$, which is in turn $\cong \ker(x)$ by $(*)$. Thus, we have $\operatorname{coker}(x) \cong \ker(x)$. It is well known from the work of Ehrlich [14] that this (together with $x = xyx$) implies that $x \in \operatorname{ureg}(R)$.

(1) $\Rightarrow$ (3). We continue to use the representation of R as an endomorphism ring $\operatorname{End}(M_k)$, and assume here that $x \in \operatorname{ureg}(R)$. By Ehrlich's result and $(*)$ above, we have $\ker(x) \cong \operatorname{coker}(x) \cong \ker(xy)$. Fix an isomorphism from $\ker(xy)$ to $\ker(x)$, and extend it to an automorphism u of M by specifying that, on $\operatorname{im}(x)$, u is the inverse of the isomorphism $\operatorname{im}(yx) \to \operatorname{im}(x)$ defined by x. Then $u \in \mathrm{U}(R)$. For any $m \in M$,

$ux(m) = ux(yx(m)) = yx(m)$, so $ux = yx \in R$. We have also $xu = xy$, since both sides are zero on $\ker(xy)$, and on a typical element $m' = x(m) \in \operatorname{im}(x)$, we have

$$xu(m') = m' = x(m) = xy(x(m)) = xy(m').$$

For later reference (see, e.g., 7.3 and footnote 4), it is of interest also to give a purely ring-theoretic argument for (1) ⇒ (3) that avoids the use of endomorphism rings. Given $x \in \operatorname{ureg}(R)$, write $x = xvx$, where $v \in \mathrm{U}(R)$. Since $xvx = xyx$, the elements $x(v - y)$ and $(v - y)x$ both have square zero, so we have the following two units in R:

$$s := 1 + x(y - v) \quad \text{and} \quad t := 1 + (y - v)x, \tag{7.2}$$

for which we have clearly $sx = x = xt$. Now let $u := tvs \in \mathrm{U}(R)$. Then

$$\begin{aligned} xu &= xtvs = xvs = xv\big[1 + x(y - v)\big] = xv + xvxy - (xv)^2 = xy \quad \text{and} \\ ux &= tvsx = tvx = \big[1 + (y - v)x\big]vx = vx + yxvx - (vx)^2 = yx, \end{aligned}$$

as desired.[3] □

Remark 7.3. Note that, in the second proof for (1) ⇒ (3) above, the existence of the unit u for (3) was shown *constructively*—starting from a given $v \in \mathrm{U}(R)$ such that $x = xvx$. To see how this construction works in a special case, let x be an idempotent $e \in R$, and consider any equation $e = eye$. In this case, we can take v to be 1. Writing f for the complementary idempotent $1 - e$, we have, in the notations of 7.2,

$$s := ey + f, \qquad t := ye + f, \quad \text{and hence} \quad u := (ye + f)(ey + f) = yey + f.$$

For this choice of u (which is a product of two units of order $\leqslant 2$), we have indeed $eu = e(yey + f) = eyey = ey$, and $ue = (yey + f)e = yeye = ye$. It is, however, not directly clear that u is a unit! It has a "highly non-obvious" inverse:

$$u^{-1} = s^{-1}t^{-1} = (1 + e - ey)(1 + e - ye) = 1 + e - ey - ye + ey^2e.$$

Following Hartwig and Luh [24], we say that $y \in R$ is an *inner inverse* for x if $xyx = x \in R$. Of course, in this situation, x may not be an inner inverse for y. (In fact, y may not even be regular.) However, *after replacing y by yxy, we may assume that x and y are inner inverses of each other*. Substitutions of the kind $y \mapsto yxy$ are well-known in the theory of generalized inverses, and will be used frequently in the rest of this paper.

[3] As an after-thought, we might add that this second proof is essentially gotten by applying Lemma 3.3 first to the right ideals $xvR = xyR$, and then to the left ideals $Rvx = Ryx$. This process produces two units in R. These are *different* in general, but then one makes a suitable change to render them the same!

Corollary 7.4. *Let $x, y \in R$ be inner inverses of each other. If $x \in \mathrm{ureg}(R)$, then there exists $u \in \mathrm{U}(R)$ such that*

$$xy = xu = u^{-1}y \quad \textit{and} \quad yx = ux = yu^{-1}.$$

Moreover, u is an inner inverse to x, and u^{-1} is an inner inverse to y (so, in particular, y is also unit-regular).

Proof. Let $u \in \mathrm{U}(R)$ be as in 7.1(3); that is, $xy = xu$, and $yx = ux$. Then $y = yxy = yxu$ gives $yx = yu^{-1}$. Similarly, $y = yxy = uxy$ gives $xy = u^{-1}y$. Finally, $x = xyx = xux$, and $y = yxy = yu^{-1}y \in \mathrm{ureg}(R)$. □

Remark 7.5. Note that a suitable converse holds for 7.4 as well. Under the standing assumption that $x = xyx$, a unit $u \in \mathrm{U}(R)$ satisfying $xy = xu = u^{-1}y$ can exist only if ($x \in \mathrm{ureg}(R)$ and) $y = yxy$. In fact, from $xy = xu = u^{-1}y$, we get $ux = uxyx = yx$, and hence $yxy = uxy = y$. A similar remark applies to the equations $yx = ux = yu^{-1}$.

Corollary 7.6. *Let e, f be isomorphic idempotents in a ring R, say $e = xy$, $f = yx$, where $x \in eRf$ and $y \in fRe$. If R is* IC*, then there exists a unit u such that $e = xu = u^{-1}y$ and $f = ux = yu^{-1}$ (and thus $e = u^{-1}fu$).*

Proof. We have $xyx = ex = x$, and $yxy = fy = y$. If R is an IC ring, then (by criterion (1.5)) $x \in \mathrm{ureg}(R)$, and we can apply 7.4. □

Of course, the converse of 7.6 also holds. In fact, it goes without saying that any new characterization of unit-regular elements in rings will lead to a new characterization of IC rings. For instance, in view of 7.1 ((1) ⇔ (3)), *a ring R is* IC *iff, whenever $x = xyx \in R$, xy is similar to yx in R.*

Yet another consequences of 7.1 in the spirit of 7.4 and 7.6 is the following, which is a well-known fact to experts in the theory of regular rings.

Corollary 7.7. *Suppose $x = xyx \in R$, with $xy = yx$. Then there exists $u \in \mathrm{U}(R)$ commuting with x such that $xy = xu$, and we have $x \in \mathrm{ureg}(R)$.*

A construction of the required unit u in this case can be gleaned from Ehrlich's proof [13, p. 210] for her theorem that strongly regular rings are unit-regular. In fact, if we write e for the idempotent xy and let $f = 1 - e$, then for $w = ex + f$ and $u = ey + f$, we have $uw = e^3 + f^2 = e + f = 1$. Thus, $u \in \mathrm{U}(R)$, and

$$xu = x(ey + 1 - e) = xxyy + x - xyx = xy,$$

as desired.

We close this section by giving the following application of 7.1. Note that, in the special case where $J = 0$, this result recaptures the equivalence of the conditions (1.4) and (1.5).

Proposition 7.8. *For any ideal $J \subseteq R$, the following statements are equivalent*:

(1) $\mathrm{reg}(R) \setminus J \subseteq \mathrm{ureg}(R)$.
(2) *For any idempotents $e, f \in R \setminus J$, $eR \cong fR$ implies that e is similar to f.*

Proof. (1) $\Rightarrow$ (2). Let $eR \cong fR$ as in (2). Then there exist $x \in eRf$ and $y \in fRe$ such that $e = xy$ and $f = yx$. Then $x = ex = xyx$, and $e \notin J \Rightarrow x \notin J$. By (1), $x \in \mathrm{ureg}(R)$, so (1) $\Rightarrow$ (5) in 7.1 yields the similarity between $e = xy$ and $f = yx$.

(2) $\Rightarrow$ (1). Let $x = xyx \notin J$. Then xy and yx are isomorphic idempotents. Since they do not lie in J, (2) implies that xy is similar to yx. By (5) $\Rightarrow$ (1) in 7.1, we conclude that $x \in \mathrm{ureg}(R)$. $\square$

8. Pseudo-similarity versus similarity

In the study of similarity of matrices in linear algebra, several weakened notions of similarity have been introduced, and these notions have been compared to the usual notion of similarity, over fields, division rings, and more generally, over unit-regular rings; see, e.g., [21–24], and [20]. The major theorem in this study is that a ring R is IC iff pseudo-similarity is equivalent to similarity over R: this is the criterion for IC rings given in (1.6). In this section, we shall give a streamlined treatment of this area of work. As it turns out, with the additional characterizations of unit-regular elements given in 7.1, the proof of the IC ring criterion in (1.6) is both easy and natural.

There are several different definitions of pseudo-similarity given in the literature, which can be a source of confusion for beginning researchers in this field. Thus, it behooves us to collect, in the following proposition, the various definitions that have been used, and to give a short proof for their equivalence. This can be done quite generally in the framework of multiplicative semigroups. (The pseudo-similarity definition given in the Introduction corresponds to that of $\sim_1$ below.)

Proposition 8.1. *On a multiplicative semigroup G, we define four binary relations $\sim_i$ $(1 \leqslant i \leqslant 4)$ as follows*:

$$a \sim_1 b \quad \Leftrightarrow \quad \exists x, z, w \in G \text{ such that } a = zbx,\ b = xaw, \text{ and } x = xzx = xwx.$$

$$a \sim_2 b \quad \Leftrightarrow \quad \exists x, y \in G \text{ such that } a = ybx,\ b = xay, \text{ and } x = xyx.$$

$$a \sim_3 b \quad \Leftrightarrow \quad \exists x, y \in G \text{ such that } a = ybx,\ b = xay, \text{ and } x = xyx,\ y = yxy.$$

$$a \sim_4 b \quad \Leftrightarrow \quad \exists x, y \in G \text{ such that } a = ybx,\ b = xay, \text{ and } (xy)^2 = xy.$$

These four relations are the same, and in particular, in view of the definition of $\sim_3$, they are symmetric relations on the semigroup G.

Proof. Clearly, $a \sim_3 b \Rightarrow a \sim_2 b \Rightarrow a \sim_1 b$. Now assume $a \sim_1 b$, and let x, z, w be as in the definition for $\sim_1$ above. Setting $y := zxw$, we check easily that x, y satisfy the

equations in the definition for $\sim_3$. Hence, $a \sim_3 b$. This proves the equality of the first three relations. (This argument is a considerable simplification of an argument given by Chen in [11, Lemma 6].)

Next, note that $a \sim_2 b \Rightarrow a \sim_4 b$ (since $xyx = x$ implies that $xyxy = xy$). Conversely, assume that $a \sim_4 b$, and let $x, y \in G$ be such that $a = ybx$, $b = xay$, and $(xy)^2 = xy$. Setting $x' = xyx$ and $y' = yxy$, we check easily that $y'bx' = a$, $x'ay' = b$, and

$$x'y'x' = (xyx)(yxy)(xyx) = (xy)^4x = xyx = x'.$$

Therefore, $a \sim_2 b$. This shows that all four relations are equal. □

If the semigroup G above is a *monoid*, then we can define conjugation (by units) in G, and thus similarity makes sense. Let us denote the similarity (equivalence) relation in this case simply by $\sim$: $a \sim b$ iff $a = u^{-1}bu$ for some unit $u \in G$. As we have observed in the Introduction, $a \sim b$ always implies $a \sim_1 b$ (and hence $a \sim_i b$ for all i). In view of this, it is of interest to ask whether (or when) the converse holds. A couple of positive cases quickly come to mind.

Examples 8.2.

(1) If, in the definition for $a \sim_i b$ ($i \leqslant 3$), the element x happens to be (left and right) cancellable in the monoid G, then z (respectively y) must be its inverse, and we have $a = x^{-1}bx \sim b$.

(2) In the definition for $a \sim_i b$ ($i \leqslant 4$), suppose the element x lies in the center of G. Then, we must have $a = b$! Indeed, for $i = 1$, we have

$$b = xaw = awx = (zbx)wx = zbx = a,$$

and the cases $i = 2, 3$ are similar (with y replacing z, w). For $i = 4$, let e be the idempotent $xy = yx$. Then

$$a = ybx = yxayx = eae, \qquad b = xay = xybxy = ebe,$$

and thus $b = e(xay)e = eaxye = eae = a$.

In general, of course, pseudo-similarity is weaker than similarity in monoids—and even in rings. This is clear, for instance, by the criterion 1.6 for IC rings: in a ring R that is not IC, pseudo-similarity fails to imply similarity in R.

While looking for conditions that would enable us to go from pseudo-similarity to similarity, we discovered the proposition below. For maximal flexibility, we shall state this result strictly for multiplicative semigroups—thus avoiding the use of an identity or unit elements.

Proposition 8.3. *Let a, b, x, y be elements in a semigroup G such that*

$$a = ybx, \qquad b = xay, \quad \textit{and} \quad x = xyx \quad (\textit{as in the definition for } a \sim_2 b).$$

If an element $u \in G$ is such that $xy = xu$ and $yx = ux$, then $au = ub$. (In particular, $ay = yb$, and we also have $bx = xa$.)

Proof. Since $a = ybx = yxayx$ and yx is an idempotent, the element a is unchanged by left and right multiplications by yx. Similarly, b is unchanged by left and right multiplications by xy. Therefore,

$$uxa = yxa = a \quad \text{and} \quad xau = x(ayx)u = (xay)xy = bxy = b.$$

From these, we get $ub = u(xau) = (uxa)u = au$. Applying this to the special case $u = y$, we get $yb = ay$, and the other equation $xa = bx$ can be proved similarly. □

The use of 8.3 should be clear to the reader! Let us now give a quick proof for the following result, which extends (1) ⇒ (8) in Theorem 2B of Hartwig and Luh [24].

Theorem 8.4. *For any ring R, let a, b, w, x, y, z be as in the definition of $a \sim_i b$ in* 8.1, *where $i \in \{1, 2, 3, 4\}$. If $x \in \mathrm{ureg}(R)$, then $a \sim b$ in R.*

Proof. For $i = 2$, this follows by combining 7.1 ((1) ⇒ (3)) and 8.3 (applied to the semigroup $(R, \cdot)$). The conclusion $au = ub$ in 8.3 implies that $a \sim b$ since the element u found in 7.1(3) is a unit.

After treating the case $i = 2$, the case $i = 3$ becomes a tautology. We can then get the case $i = 1$ too, since the transition from $a \sim_1 b$ to $a \sim_3 b$ involves only the transformation $y = zaw$, *with no change in* x. Finally, to get the case $i = 4$, recall that the transition from $a \sim_4 b$ to $a \sim_2 b$ involves the transformations $x' = xyx$ and $y' = yxy$. The change in y is harmless, but we must check that the hypothesis $x \in \mathrm{ureg}(R)$ is preserved under the transformation $x \mapsto x'$. Now if v is any unit such that $x = xvx$, then

$$x'vx' = (xyx)v(xyx) = (xy)x(yx) = (xy)^2x = xyx = x',$$

so we still have $x' \in \mathrm{ureg}(R)$, as desired. □

Example 8.5. It is pertinent to point out that Theorem 8.4 is strictly a ring-theoretic result. Its proof depends on the existence of a special unit $u \in \mathrm{U}(R)$, which was constructed by using addition and subtraction in a ring, as well as multiplication. In a monoid G, unit-regular elements still make sense, but the statement in 8.4 may no longer be true. A minimal counterexample can be produced easily as follows. Let R be a ring with two distinct nontrivial idempotents e, f such that $Re = Rf$. Then $ef = e$, $fe = f$, and $G := \{1, e, f\}$ is a 3-element monoid. If we define $a = y = e$ and $b = x = f$, then $a = ybx$, $b = xay$, $x = xyx$, and $y = yxy$ all hold, so $a \sim_3 b$, with x unit-regular (it is an idempotent f). However, $a = e$ is not similar to $b = f$ in G, since the unit group of G is trivial. To produce a "bigger" counterexample, let $R = \mathbb{M}_2(\mathbb{Z})$, and let S be the semiring of matrices in R with non-negative entries. Then $e := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $f := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ satisfy $Re = Rf$, so as before, $e \sim_3 f$ in S. However, e, f are still not similar in S, since the unit group of S consists of the identity and $\sigma = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and σ does not conjugate e to f. This shows that 8.4

does not hold even for semirings with identity.[4] But of course, e and f are similar in the ring R; after all, R is an IC ring by 5.7.

With Theorem 8.4 at our disposal, it is now a routine matter to deduce the criterion 1.6 for IC rings, due to Guralnick–Lanski [20], and Hall, Hartwig, Katz, and Neuman [22].

Theorem 8.6. *A ring R is* IC *iff pseudo-similarity implies similarity in R.*

Proof. The "only if" part follows from 8.4 since $\mathrm{reg}(R) = \mathrm{ureg}(R)$ in an IC ring. Conversely, suppose pseudo-similarity implies similarity in R. Consider any pair of isomorphic idempotents $a, b \in R$. Write $a = yx$ and $b = xy$, where $y \in aRb$ and $x \in bRa$. Then

$$ybx = yx = a, \quad xay = xy = b, \quad xyx = xa = x, \quad \text{and} \quad yxy = ay = y.$$

Thus, $a \sim_3 b$, which implies that $a \sim b$. By the criterion 1.4, R is an IC ring. (Note that we have actually proved more than is required. The argument for the "if" part would work here as long as pseudo-similarity implies similarity *for idempotents* in R.) □

As another application of the results in Sections 7–8, let us also record the following curious criterion for similarity in an arbitrary ring.

Theorem 8.7. *Two elements a, b in a ring R are similar if and only if there exist two commuting elements $x, y \in R$ such that xy is an idempotent and $a = ybx$, $b = xay$.*

Proof. The "necessity" part is trivial: if $a = u^{-1}bu$ for some $u \in \mathrm{U}(R)$, we are done by choosing $x = u$ and $y = u^{-1}$. For the "sufficiency" part, assume that $a = ybx$ and $b = xay$, with $xy = yx$ an idempotent. Then we have $a \sim_4 b$, although we cannot apply 8.4 right away since we may not have $x \in \mathrm{ureg}(R)$. However, if we make the (by now familiar) transformations $x' = xyx$ and $y' = yxy$, we get

$$a = y'bx', \qquad b = x'ay', \quad \text{and} \quad x'y'x' = x', \tag{8.8}$$

which attest to $a \sim_2 b$. Now, $xy = yx \Rightarrow x'y' = y'x'$, so 7.7 implies that $x' \in \mathrm{ureg}(R)$. Thus, by 8.4 (applied to $a \sim_2 b$ via (8.8)), we conclude that $a \sim b$ in R. □

Corollary 8.9. *Let r be an element in a ring R such that $e := r^n$ is an idempotent* (*for some $n \geqslant 1$*). *Then, for any $a \in eRe$, a is similar to $r^i a r^j$ whenever $i + j = n$.*

Proof. Let $x = r^i$, $y = r^j$, and $b = xay$. Then $ybx = r^j(r^i a r^j)r^i = eae = a$, and $xy = yx = r^n = e$ is an idempotent. Therefore, by 8.7, a is similar to $b = r^i a r^j$. □

[4] In retrospect, we note that, in the construction of the crucial unit $u = tvs$ in the second proof of (1) ⇒ (3) in 7.1, a single minus sign was used in producing each of the units s and t in 7.2!

9. Results on lifting ideals

In this final section, we will show that, to check the characterizing property $\mathrm{reg}(R) \subseteq \mathrm{ureg}(R)$ of an IC ring, we may "ignore" elements in a given proper ideal $J \subsetneq R$ as long as J has a certain "lifting property": see Theorem 9.7. This result is inspired by Chen's Theorem 2.2 in [8], which states that, if J is a proper ideal in an exchange ring R, then R has stable range 1 iff every regular element in $R \setminus J$ is unit-regular. Here, we work more generally with the characterization of IC rings; Chen's theorem can be recaptured by simply specializing our result to exchange rings: see Corollary 9.8.

To formulate our approach more precisely, recall that an element $x \in R$ is said to be an idempotent modulo a left ideal $I \subseteq R$ if $x - x^2 \in I$. In this case, we say x can be lifted to an idempotent (modulo I) if there exists an idempotent $e \in R$ such that $e - x \in I$. (This terminology comes from Nicholson's fundamental paper [38].) Taking these ideas one step further, let us say that an ideal $J \subseteq R$ is a *lifting ideal* if idempotents lift modulo every left ideal I contained in J. For instance, nil ideals J are always lifting, as we can easily check by a slight modification of the standard argument given in [29, (21.28)]. According to [38, Corollary 1.3], R is an exchange ring iff R is a lifting ideal of itself. More generally, using Proposition 1.1 in [38], it is easy to show that an ideal J is a lifting ideal in a ring R iff the following equivalent properties hold:

(9.1) *If $x \in R$ is such that $x^2 - x \in J$, then there exists an idempotent $e \in Rx$ such that $1 - e \in R(1-x)$.*
(9.2) *If $x \in R$ is such that $x^2 - x \in J$, then there exists an idempotent $e \in xR$ such that $1 - e \in (1-x)R$.*

(See Lemma 2.3 and Proposition 4.3 in [28].) In particular, it follows that J being a lifting ideal is a left/right symmetric notion.

We should point out that, in checking that J is a lifting ideal, it is *not* enough to check that idempotents lift modulo J itself. For instance, in the ring $\mathbb{Z}$, idempotents can certainly be lifted modulo $5\mathbb{Z}$, but $5\mathbb{Z}$ is not a lifting ideal since idempotents cannot be lifted modulo the subideal $10\mathbb{Z} \subseteq 5\mathbb{Z}$.

In a similar vein, we can define the notion of $x \in R$ being regular modulo a left ideal I (meaning $x - xyx \in I$ for some $y \in R$), and being liftable to a regular element modulo I (meaning $z - x \in I$ for some $z \in \mathrm{reg}(R)$). With these definitions in place, we shall prove the following result on lifting regular elements extending [16, Corollary 5]. The proof here proceeds along the same lines as that of Theorem 3.2.1 in the thesis of I. de las Peñas. We thank Professor E. Sánchez Campos for communicating to us the latter proof.

Theorem 9.3. *Let J be any lifting ideal in a ring R. Then regular elements lift modulo every left ideal $I \subseteq J$.*

Proof. Let $x, y \in R$ be such that $x - xyx \in I \subseteq J$. Then $xyxy - xy = (xyx - x)y \in J$, so by 9.1 there exists an idempotent $f \in Rxy$ such that $1 - f \in R(1-xy)$. Write $f = rxy$ and $1 - f = s(1-xy)$, where $r, s \in R$. Setting $z := frx$, we have $zy = frxy = f^2 = f$. Thus, $zyz = fz = z$, so $z \in \mathrm{reg}(R)$, and

$$\begin{aligned} z - x &= z(1-yx) - (1-zy)x = frx(1-yx) - (1-f)x \\ &= frx(1-yx) - s(1-xy)x = (fr-s)(x-xyx) \in I. \end{aligned}$$

This shows that the element $z \in \operatorname{reg}(R)$ "lifts" the given x modulo I, as desired. □

In the case where R is an exchange ring, the hypothesis on J in 9.3 is automatically satisfied. In this case, 9.3 says that regular elements lift modulo every left ideal in R: this is essentially [16, Corollary 5], which 9.3 purports to generalize. Note that the assumption that J be a lifting ideal is crucial for 9.3, as the same result fails to hold if we only assume that idempotents can be lifted modulo J. For instance, for $J = 5\mathbb{Z}$ in the ring $\mathbb{Z}$, idempotents obviously lift modulo J, but the element $\bar{2}$ is invertible and hence regular in $\mathbb{Z}/5\mathbb{Z}$, and does not lift to a regular element in $\mathbb{Z}$. What goes wrong here is that J is *not* a lifting ideal, as we have pointed out earlier.

There is however, one case where it is enough to assume that idempotents can be lifted modulo J. This is the case where J is inside the Jacobson radical. In this case, J is a lifting ideal as long as idempotents can be lifted modulo J: see Theorem 2.4 and Remark 2.5 in [28]. In view of this, we have the following pleasant consequence of 9.3.

Corollary 9.4. *If idempotents can be lifted modulo a given ideal $J \subseteq \operatorname{rad}(R)$, then regular elements also lift modulo J.*

For the purpose of proving our main result 9.7 below, we shall need two lemmas, the first of which is the following fact concerning the contraction of a lifting ideal to a Peirce corner ring eRe.

Lemma 9.5. *Let J be any lifting ideal in a ring R. Then for any idempotent $e \in R$, the contraction $J \cap eRe = eJe$ is a lifting ideal in the corner ring eRe.*

Proof. Let $x \in eRe$ be such that $x^2 - x \in eJe$. To check 9.1 for the pair $eJe \subseteq eRe$, we need to show that there is an idempotent $g \in eRe \cdot x$ such that $e - g \in eRe \cdot (e - x)$. As $x^2 - x \in eJe \subseteq J$, 9.1 (for the pair $J \subseteq R$) gives an idempotent $f \in Rx$ such that $1 - f \in R(1-x)$. Since

$$x \in eRe \quad \Rightarrow \quad fe = f \quad \Rightarrow \quad (ef)^2 = efef = ef,$$

ef is an idempotent in eRe. Now

$$e - ef = e(1-f)e \in eR(1-x)e = eRe \cdot (e - x),$$

which checks the validity of 9.1 for the pair $eJe \subseteq eRe$. □

Lemma 9.6. *Let J be any lifting ideal in a ring R, and let $\overline{R} := R/J$. If e, f are idempotents in R, then $\bar{e}\overline{R} \cong \bar{f}\overline{R}$ iff there exist right ideal decompositions $eR = A_1 \oplus A_2$ and $fR = B_1 \oplus B_2$ such that $A_1 \cong B_1$ and $A_2 = A_2 J$, $B_2 = B_2 J$.*

Proof. The proof below is extracted from that of Proposition 1.4 in [2]. If the said decompositions exist, then

$$\bar{e}\overline{R} \cong eR/eJ = (A_1 \oplus A_2)/(A_1 J \oplus A_2 J) \cong A_1/A_1 J,$$

and similarly $\overline{f}\overline{R} \cong B_1/B_1 J$. Therefore, we have $\bar{e}\overline{R} \cong \overline{f}\overline{R}$.

Conversely, if this isomorphism holds, there exist $x \in eRf$ and $y \in fRe$ such that $xy \equiv e \pmod{J}$ and $yx \equiv f \pmod{J}$. From

$$xy \in eRe \quad \text{and} \quad (xy)^2 - xy \in J \cap eRe = eJe,$$

we get from 9.5 an idempotent $g \in xy \cdot eRe$ such that $e - g \in (e - xy) \cdot eRe \subseteq J$. Let $g = xyt$, where $t \in eRe$. As $g^2 = g$, we may assume that $t \in eReg \subseteq eRg$. Then $g - et = (xy - e)t \in J$, so $e \equiv g \equiv et = t \pmod{J}$. Setting $h := ytx \in fRf$, we have

$$h^2 = ytxytx = ytgx = ytx = h,$$

so h is an idempotent, with $hR \cong gR$. Now $e - t \in J$ gives

$$f \equiv yx = yex \equiv ytx = h \pmod{J}.$$

Now from $g \in eRe$ and $h \in fRf$, we have the decompositions

$$eR = gR \oplus (e - g)R \quad \text{and} \quad fR = hR \oplus (f - h)R.$$

We are done by setting

$$A_1 = gR, \qquad A_2 = (e - g)R, \qquad B_1 = hR, \quad \text{and} \quad B_2 = (f - h)R,$$

since (as noted above) $gR \cong hR$, and the fact that $e - g$ and $f - h$ are idempotents in J implies that $A_2 = A_2 J$ and $B_2 = B_2 J$. □

We can now prove the main result in this section on certain "modified" characterizations of IC rings. Recall that the symbol "$\sim$" denotes similarity between elements in a ring.

Theorem 9.7. *Given a lifting ideal J in a ring R, consider the following conditions*:

(1) *R is an* IC *ring*;
(2) *$\overline{R} := R/J$ is* IC, *and for any idempotents $e, f \in 1 + J$, $eR \cong fR \Rightarrow e \sim f$*;
(3) *for any idempotents $e, f \in R \setminus J$, $eR \cong fR \Rightarrow e \sim f$*;
(4) $\mathrm{reg}(R) \setminus J \subseteq \mathrm{ureg}(R)$.

In general, (1) $\Leftrightarrow$ (2), *and if $J \neq R$, these are also equivalent to* (3) *and* (4).

Proof. (1) $\Rightarrow$ (2). Assuming (1), the second half of (2) follows tautologically from 1.4. For the first half, let $\bar{x} \in \mathrm{reg}(\overline{R})$. By 9.3, we may assume that $x \in \mathrm{reg}(R)$. Then $x \in \mathrm{ureg}(R)$ by 1.5, and so $\bar{x} \in \mathrm{ureg}(\overline{R})$.

(2) $\Rightarrow$ (1). This implication is inspired by Chen's result in [8, Lemma 2.1], and its proof runs along similar lines (although we do not assume R to be an exchange ring as Chen did). To show that R is IC, we start with $eR \cong fR$ (where e, f are *arbitrary* idempotents in R), and try to check that $e \sim f$ (or equivalently, that $(1-e)R \cong (1-f)R$). Since $\bar{e}\overline{R} \cong \bar{f}\overline{R}$, the first half of (2) implies that $\overline{(1-e)}\overline{R} \cong \overline{(1-f)}\overline{R}$. Applying 9.6, we have right ideal decompositions

$$(1-e)R = A_1 \oplus A_2 \quad \text{and} \quad (1-f)R = B_1 \oplus B_2$$

such that $A_1 \cong B_1$ and $A_2 = A_2 J$, $B_2 = B_2 J$. Since A_2, B_2 are direct summands of R_R contained in J, there exist idempotents $g, h \in J$ such that $A_2 = gR$ and $B_2 = hR$. From

$$R = (1-g)R \oplus gR = eR \oplus A_1 \oplus gR,$$

we see that $(1-g)R \cong eR \oplus A_1$, and similarly, $(1-h)R \cong fR \oplus B_1$. Therefore, $(1-g)R \cong (1-h)R$. Since $1-g, 1-h \in 1+J$, the second half of (2) implies that $gR \cong hR$, and hence

$$(1-e)R = A_1 \oplus A_2 \cong B_1 \oplus B_2 = (1-f)R,$$

as desired.

Next, (1) $\Rightarrow$ (4) follows from 1.5, and (3) $\Leftrightarrow$ (4) is true for *any* ideal $J \subseteq R$ by 7.8. To complete the proof, we need only show that (3) and (4) (together) imply (2) in the case $J \neq R$. Assuming $J \neq R$, (3) certainly implies the second half of (2), since $1 + J \subseteq R \setminus J$. To show the first half of (2), it suffices to check that $\mathrm{reg}(\overline{R}) \subseteq \mathrm{ureg}(\overline{R})$. Let $0 \neq \bar{x} \in \mathrm{reg}(\overline{R})$. In view of 9.3, we may assume that $x \in \mathrm{reg}(R)$. As $\bar{x} \neq 0$, $x \in R \setminus J$. So by (4), $x \in \mathrm{ureg}(R)$, and thus $\bar{x} \in \mathrm{ureg}(\overline{R})$. □

Specializing 9.7 to exchange rings, we can now retrieve the following result of Chen [8, Theorem 2.2]. (In Chen's result, however, the crucial hypothesis that $J \neq R$ was left out.)

Corollary 9.8. *Let J be a proper ideal of an exchange ring R. Then the following are equivalent*:

(1) *R is an* IC *ring*;
(2) *R has stable range* 1;
(3) *for any idempotents $e, f \in R \setminus J$, $eR \cong fR \Rightarrow e \sim f$*;
(4) $\mathrm{reg}(R) \setminus J \subseteq \mathrm{ureg}(R)$.

Proof. Since R is an exchange ring, J is automatically a lifting ideal, so the result follows from 6.5 and 9.7. □

Appendix. Failure of cancellation over group algebras

By R.G. Swan

In this appendix, we offer an example of a complex group algebra $\mathbb{C}G$ that is not an IC ring. More precisely we will prove the following.

Theorem A.1. *There is a group G whose complex group algebra $A = \mathbb{C}G$ has the following properties*:

(I) *There is a right A-module P with $P \oplus A \cong A^2$ but $P \ncong A$.*
(II) *A is not* IC; *that is, there are right ideals I, J, I', J' with $A = I \oplus J = I' \oplus J'$, $I \cong I'$ but $J \ncong J'$.*

Remark A.2.

(a) If a ring A has either of these properties, so does $A \times B$ for any ring B. For property (I) we use $P \oplus B$ while for property (II) we use $I \oplus B$, $I' \oplus B$, J, and J'.
(b) If a ring A has property (I), then $\mathbb{M}_2(A)$ has property (II). (Property (I) on A implies the failure of internal cancellation on A^2_A, so $\mathrm{End}(A^2_A) \cong \mathbb{M}_2(A)$ is not IC.)

The proof of 5.10 above already made use of the theorem of Ojanguren and Sridharan from [39]. To prove Theorem A.1, we shall use the following variant of the Ojanguren–Sridharan result.

Theorem A.3. *Let D be a domain having elements a and b such that $u = ab - ba$ is a unit. Then the ring $R = D[x, x^{-1}, y, y^{-1}]$ has property* (I) *of Theorem* A.1.

As in [39], the module P is defined to be the kernel of $R^2 \to R$ sending (ξ, η) to $(x + a)\xi + (y + b)\eta$. The corresponding result for the ring $D[x, y]$ is proved in [44, Lemma 3.1]. Theorem A.3 follows immediately from the case of $D[x, y]$ and the following lemma. As usual, for a central element $s \in R$ we write R_s for $R[s^{-1}]$ and P_s for $P \otimes_R R[s^{-1}]$.

Lemma A.4. *Let D be a domain and let P be a submodule of a free right module $R^{(N)}$ over the polynomial ring $R = D[x_1, \ldots, x_n]$. If $P_{x_1 \cdots x_n} \cong R_{x_1 \cdots x_n}$ then $P \cong R$.*

Proof. We may assume that N is finite because $P_{x_1 \cdots x_n}$, being finitely generated, lies in a finitely generated free summand and therefore so does P. We can find an element $(f_1, \ldots, f_N)$ of P such that

$$P_{x_1 \cdots x_n} = (f_1, \ldots, f_N) \cdot R_{x_1 \cdots x_n}.$$

Choose this so $\sum \deg f_i$ is minimal. Then no x_i divides all f_j otherwise we could replace each f_j by f_j/x_i reducing the degree. If $(g_1, \ldots, g_N) \in P$ we can find $e_i \geqslant 0$ such that

$$(g_1, \ldots, g_N) \cdot x_1^{e_1} \cdots x_n^{e_n} = (f_1, \ldots, f_N) \cdot h$$

for some $h \in R$. Choose this so $\sum e_i$ is least. If $e_i > 0$, choose j so x_i does not divide f_j. Since x_i divides $f_j h$ and $R/(x_i)$ is a domain, x_i must divide h and we can reduce e_i by using h/x_i in place of h. This shows that all e_i must be 0 and therefore $P = (f_1, \ldots, f_N) \cdot R \cong R$. $\square$

We shall now give an example of a group G for which $\mathbb{C}G$ has property I of Theorem A.1. Let H be the group generated by elements a, b, and c, with the relations

$$c = [a, b], \qquad [a, c] = 1, \qquad [b, c] = 1, \quad \text{and} \quad c^2 = 1.$$

Then c generates a central subgroup C of order 2, and H/C is free abelian on two generators. Let $e = (1 + c)/2$ in $\mathbb{C}H$. Then e is a central idempotent so $\mathbb{C}H = D' \times D$ where $D' = e \cdot \mathbb{C}H$ and $D = (1 - e) \cdot \mathbb{C}H$. To determine these two rings, note that in $D' \cong \mathbb{C}H/D$, e is "equated" to 1 and hence so is c. Thus, $D' \cong \mathbb{C}[a, a^{-1}, b, b^{-1}]$. Similarly, in $D \cong \mathbb{C}H/D'$, e is equated to 0 and c to -1, so D is isomorphic to the twisted Laurent polynomial ring over $\mathbb{C}[a, a^{-1}]$ generated by b with $ba = -ab$. In particular, D is also a domain. Since $u = ab - ba = 2ab$ is a unit, Theorem A.3 applies to D. Let F be a free abelian group on two generators x and y and let $G = H \times F$. Then

$$\mathbb{C}G \cong \mathbb{C}H \otimes_{\mathbb{C}} \mathbb{C}F \cong D'[F] \times D[F].$$

Since $D[F] = D[x, x^{-1}, y, y^{-1}]$, it has property (I) and therefore so does $\mathbb{C}G$ by Remark A.2(a).

With the above group G at our disposal, Theorem A.1 can now be deduced from the following result.

Proposition A.5. *Let G be any group such that $\mathbb{C}G$ has property* (I) *of Theorem* A.1. *Let S be the symmetric group on three letters. Then $\mathbb{C}[G \times S]$ has properties* (I) *and* (II) *of Theorem* A.1.

Proof. It is well known that $\mathbb{C}S \cong \mathbb{C} \times \mathbb{C} \times \mathbb{M}_2(\mathbb{C})$. Therefore

$$\mathbb{C}[G \times S] \cong \mathbb{C}G \otimes_{\mathbb{C}} \mathbb{C}S \cong \mathbb{C}G \times \mathbb{C}G \times \mathbb{M}_2(\mathbb{C}G).$$

Applying Remark A.2(a) to the first factor shows that $\mathbb{C}[G \times S]$ has property (I), and applying *both* parts of Remark A.2 to the last factor shows that $\mathbb{C}[G \times S]$ has property (II). $\square$

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