Determinantal Ideals, Pfaffian Ideals, and the Principal Minor Theorem

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Abstract. This paper is devoted to the study of determinantal and Pfaffian ideals of symmetric/skew-symmetric and alternating matrices over general commutative rings. We show that, over such rings, alternating matrices A have always an *even* McCoy rank by relating this rank to the principal minors of A. The classical Principal Minor Theorem for symmetric and skew-symmetric matrices over fields is extended to a certain class of commutative rings, as well as to a new class of *quasi-symmetric* matrices over division rings. Other results include a criterion for the linear independence of a set of rows of a symmetric or skew-symmetric matrix over a commutative ring, and a general expansion formula for the Pfaffian of an alternating matrix. Some useful determinantal identities involving symmetric and alternating matrices are also obtained. As an application of this theory, we revisit the theme of alternating-clean matrices introduced earlier by two of the authors, and determine all diagonal matrices with non-negative entries over $\mathbb{Z}$ that are alternating-clean.

1. Introduction

Following Albert [Al], we call a square matrix A *alternate* (or more popularly, *alternating*) if A is skew-symmetric and has zero elements on the diagonal. The set of alternating $n \times n$ matrices over a ring R is denoted by $\mathbb{A}_n(R)$. In the case where R is commutative and n is even, the most fundamental invariant of a matrix $A \in \mathbb{A}_n(R)$ is its *Pfaffian* $\mathrm{Pf}\,(A) \in R$ (see [Ca], [Mu: Art. 412], or [Ar: III.3.27]), which has the property that $\det(A) = (\mathrm{Pf}\,(A))^2$. In order that this equation holds for alternating matrices of all sizes, we define $\mathrm{Pf}\,(A)$ to be 0 when n is odd.

In [Al], Albert developed the basic algebraic theory of symmetric, skew-symmetric, and alternating matrices over a field. As most of Albert's results depended on the field assumption, it was not clear how much of this theory can be carried over to matrices with entries from a commutative ring with possibly 0-divisors. In this case, symmetric and alternating matrices correspond, respectively, to symmetric bilinear forms and symplectic forms on free modules. Such structures — even on projective modules — are certainly of importance over commutative rings in general; see, e.g. [La$_4$: Ch. 7]. In commutative algebra, the study of determinantal

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ideals associated with the generic matrix is motivated by their connections with algebraic geometry, combinatorics, invariant theory, and representation theory (see [BC] and [BV] for surveys and literature). Much of this work is concerned with the projective resolutions of the determinantal ideals in the generic case, and with the study of the algebraic and geometric properties of the determinantal rings defined by these ideals. The case of symmetric and skew-symmetric matrices has been extensively studied also; see [He], [Kut], [Ku], [Jo], [JP], [JPW], and [HoS] for some representative works, and [Co] (and its predecessor [St]) for connections to computer algebra and Gröbner bases. Most of the papers cited above are, however, closely tied to the case of the generic matrix; a general algebraic study of the properties of the determinantal ideals of a symmetric or skew-symmetric matrix over an *arbitrary* commutative ring has remained relatively lacking.

In this paper, we focus our attention on the study of such determinantal ideals as well as the Pfaffian ideals of alternating matrices over a commutative ring R. A key tool in our study is the notion of the *McCoy rank* of a rectangular matrix A over R, which so far seemed to have escaped notice in the research on symmetric and skew-symmetric matrices. For the reader's convenience, we recall this notion in §2. Using this notion, we can define a square matrix $A \in \mathbb{M}_n(R)$ to be *nonsingular* if $\mathrm{rk}\,(A) = n$ (and *singular* otherwise). Since other notions such as linear combinations, linear dependence and independence, etc. are all meaningful over R, it becomes possible to formulate and to solve various linear algebra and matrix-theoretic problems for the class of symmetric/skew-symmetric and alternating matrices over a commutative ring R.

As we have indicated in the opening paragraph, an alternating matrix of odd size has a zero determinant (see [Ca]). In Theorem 3.2, we use this property to show that any alternating matrix $A \in \mathbb{A}_n(R)$ has *even* (McCoy) rank, and that the $2k$-th and $(2k-1)$-st determinantal ideals of A have always the same radical in R. Various other elementary relations between these determinantal ideals and the associated Pfaffian ideals (generated by the Pfaffians of the principal submatrices of A of different sizes) are also developed in §3.

In the matrix theory over fields, one of the best known classical results about symmetric and skew-symmetric matrices A of rank r is the *Principal Minor Theorem,* which states that A has a nonzero $r \times r$ principal minor; in other words, some $r \times r$ principal submatrix of A must be nonsingular. In §2, we study this result from a new perspective, and generalize it to a broader class of matrices — called *quasi-symmetric matrices* — over an arbitrary division ring. The resulting generalization, Theorem 2.9, represents a strong form of the Principal Minor Theorem, to the effect that, for a quasi-symmetric matrix A of rank r over a division ring, the principal submatrix sitting on any r left linearly independent rows of A is nonsingular (and conversely). This result applies well, for instance, to the class of square matrices satisfying the null space condition of Horn and Sergeichuk [HS] in the setting of fields with involutions.

In the second half of Section 3, we study possible generalizations of the Principal Minor Theorem to symmetric and skew-symmetric matrices over a commutative ring R. Caution must be exercised here, since in general a matrix $A \in \mathbb{M}_n(R)$ of rank r *need not* have a nonsingular $r \times r$ submatrix, or a set of r linearly independent rows. Thus, the classical form of the Principal Minor Theorem cannot be expected to hold verbatim for the class of symmetric and skew-symmetric matrices over R. However, the "obstruction" for finding a nonsingular submatrix of size $\mathrm{rk}\,(A)$ (for all A) can be explicitly identified. It turns out to hinge upon the following condition:

$(*)$: *Any finite set of* 0*-divisors in* R *has a nonzero annihilator.*

In Thm. 3.15, we show that, if R has the property $(*)$, then both forms of the Principal Minor Theorem hold for any symmetric or skew-symmetric matrix over R. The rings R for which Thm. 3.15 is applicable include, for instance, all noetherian rings whose 0-divisors happen to form an ideal, and all rings of the form $S/\mathfrak{q}$ where $\mathfrak{q}$ is a primary ideal in a ring S.

Section 4 is devoted to the proof of a certain combinatorial Expansion Formula (4.3) for computing the Pfaffian of an alternating matrix, which generalizes the classical formula for the row expansion of the Pfaffian. The proof of this Expansion Formula is based on a counting technique, which reduces the argument to a case of Newton's Binomial Theorem for negative integral exponents.

In §5, utilizing a corollary of Theorem 2.9, we obtain a criterion for a given set of p rows of a symmetric or skew-symmetric A to be linearly independent, in terms of the density of an ideal generated by a suitable set of principal minors of A of size up to and including $2p$. For instance, if all such principal minors are nilpotent (and $R \neq 0$), then the given p rows of A are linearly dependent over R.

In §6, we prove two determinantal identities (6.3) and (6.8), with the goal of applying them to the study of the alternating-clean matrices introduced earlier by two of the authors in [LS]. A matrix $M \in \mathbb{M}_n(R)$ is said to be *alternating-clean* (or $\mathbb{A}$-*clean* for short) if $M = A + U$ where A is alternating and U is invertible. The classification of such matrices M over fields was completed in [LS], but the study of $\mathbb{A}$-clean matrices over commutative rings R is considerably harder. The determinantal identity in (6.3) implies, for instance, that any *even-sized* symmetric matrix over R with all 2×2 minors zero is $\mathbb{A}$-clean.

The last two sections of the paper are concerned with the study of *diagonal* matrices that are $\mathbb{A}$-clean. After giving some examples of such matrices in §7 via the determinantal identity (6.8), we focus on the case of diagonal matrices with non-negative integer entries, and give in Theorem 8.1 a complete classification for such matrices that are $\mathbb{A}$-clean. The crux of the proof for this classification is the Vanishing Theorem 4.4, which is an easy consequence of the Pfaffian Expansion Formula in (4.3). In case it may not be apparent to our readers, we should like to point out that it was originally the desire for proving Theorem 8.1 that had first prompted us to undertake a more general study of the Pfaffian ideals of an alternating matrix. The latter study directly led to this work.

Throughout this paper, R *denotes a commutative ring with* 1. On a few occasions (mainly in §2), we also deal with possibly noncommutative rings; we shall denote these by K. The additive group of $m \times n$ matrices over K is denoted by $\mathbb{M}_{m,n}(K)$, and we write $\mathbb{M}_n(K)$ for the ring $\mathbb{M}_{n,n}(K)$. The transpose of a matrix M is denoted by M^T. For a commutative ring R, $\mathcal{Z}(R)$ denotes the set of 0-divisors in R, with the convention that $0 \in \mathcal{Z}(R)$. The notation $I \lhd R$ means that I is an ideal in R, and we write $\mathrm{rad}\,(I)$ for the radical of I. Other notations used in the paper are mostly standard; see, e.g. [Ka].

2. Matrix Ranks over Commutative Rings and Division Rings

For an $m \times n$ matrix A over a commutative ring R, let $\mathcal{D}_i(A)$ denote its i-th determinantal ideal; that is, the ideal in R generated by all $i \times i$ minors of A. It is convenient to take $\mathcal{D}_0(A) = R$, and $\mathcal{D}_i(A) = (0)$ if $i > \min\{m, n\}$, so that

$$R = \mathcal{D}_0(A) \supseteq \mathcal{D}_1(A) \supseteq \mathcal{D}_2(A) \supseteq \cdots \supseteq (0). \tag{2.0}$$

An ideal $J \lhd R$ is said to be *dense* (or *faithful*) if $\operatorname{ann}(J) = 0$. For future reference, we state the following fact, which is easily proved by induction on t.

Proposition 2.1. *For any integer $t > 0$, an ideal J is dense iff J^t is dense.*

For $R \neq 0$, the McCoy rank of A, denoted by $\operatorname{rk}(A)$, is defined to be the largest integer r such that $\mathcal{D}_r(A)$ is dense in R. In view of (2.0), $\operatorname{rk}(A)$ is the unique integer $r \geq 0$ such that $\operatorname{ann}(\mathcal{D}_r(A)) = 0$ while $\operatorname{ann}(\mathcal{D}_i(A)) \neq 0$ for all $i > r$. Clearly, the McCoy rank (henceforth simply called the rank) directly generalizes the usual rank of matrices over fields and integral domains. The following basic result of McCoy [Mc: p. 159-161] will be used freely throughout.

McCoy's Rank Theorem 2.2. *Let $R \neq 0$ and $m \leq n$. A matrix $A \in \mathbb{M}_{m,n}(R)$ has rank m iff the rows of A are linearly independent. If $m = n$, this is equivalent to $\det(A)$ being a non 0-divisor in R, and hence also equivalent to the columns of A being linearly independent. In this case, A is called* nonsingular; *otherwise, A is called* singular.

Note that "nonsingular" is much weaker than "invertible", since A invertible (written $A \in \mathrm{GL}_n(R)$) would amount to $\det(A)$ being a unit of R. The following are some other easy observations on the rank.

Remarks 2.3. (1) If B is a submatrix of A, then $\operatorname{rk}(B) \leq \operatorname{rk}(A)$.

(2) Given a matrix $A \in \mathbb{M}_{m,n}(R)$, $\operatorname{rk}(A)$ may depend on the choice of the ambient ring R. If $R \subseteq R'$ are commutative rings (with the same identity), it is easy to see that $\operatorname{rk}_{R'}(A) \leq \operatorname{rk}_R(A)$, but equality does not hold in general. For a rather extreme example, suppose $a \in R$ is a non 0-divisor in R but is a 0-divisor in R'. Then the matrix $A = a \cdot I_n$ has rank n over R, but rank 0 over R'!

(3) The rank of a matrix behaves as well as we could expect with respect to localizations. This can be seen through the following lemma, which is a well-known fact in commutative algebra.

Lemma 2.4. *A finitely generated R-module M is faithful iff its localization $M_{\mathfrak{p}}$ is faithful for every prime ideal $\mathfrak{p} \lhd R$. In particular, a determinantal ideal $I \lhd R$ is dense in R iff $I_{\mathfrak{p}}$ is dense in $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \lhd R$.*

Proposition 2.5. *Let A be any matrix over R. Then $\operatorname{rk}(A) = \min\{\operatorname{rk}(A_{\mathfrak{p}})\}$, where $\mathfrak{p}$ ranges over all prime ideals of R.*

Proof. Let $r = \operatorname{rk}(A)$. Then $\mathcal{D}_r(A)$ is dense in R, so by (2.4), $\mathcal{D}_r(A_{\mathfrak{p}})$ is dense in $R_{\mathfrak{p}}$ for every prime $\mathfrak{p} \lhd R$. This shows $\operatorname{rk}(A_{\mathfrak{p}}) \geq r$. Thus, $\min\{\operatorname{rk}(A_{\mathfrak{p}})\} \geq r$. On the other hand, since $\mathcal{D}_{r+1}(A)$ is not dense, (2.4) shows that there exists a prime $\mathfrak{p} \lhd R$ such that $\mathcal{D}_{r+1}(A_{\mathfrak{p}})$ is not dense, so $\operatorname{rk}(A_{\mathfrak{p}}) \leq r$. Thus, we also have $\min\{\operatorname{rk}(A_{\mathfrak{p}})\} \leq r$, as desired. □

It is of interest to study the notion of matrix ranks over division rings too, so let us recall it briefly. For any matrix $A \in \mathbb{M}_{m,n}(K)$ over a division ring K, $\mathrm{rk}\,(A)$ denotes the left row rank of A; that is, the dimension of the left row space of A. It is well known that this is always equal to the right column rank of A; that is, the dimension of the right column space of A. (For a couple of quick and self-contained proofs of this fact, see [La_1, La_2].) Consistently with the case of commutative rings, if $A \in \mathbb{M}_n(K)$ has rank n, we say A is *nonsingular.* Over division rings, this is *equivalent* to having $A \in \mathrm{GL}_n(K)$. Also, for any $A \in \mathbb{M}_{m,n}(K)$, $\mathrm{rk}\,(A)$ is the largest integer r such that A has a nonsingular $r \times r$ submatrix[1] [La_3: Exer. 13.14]. As is also pointed out in [La_1], in general, $\mathrm{rk}\,(A) \neq \mathrm{rk}\,(A^T)$. In particular, $A \in \mathrm{GL}_n(K)$ need not imply $A^T \in \mathrm{GL}_n(K)$. For instance, over the real quaternions, the matrix $A = \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}$ is invertible (with inverse $\frac{1}{2}\begin{pmatrix} 1 & -j \\ -i & -k \end{pmatrix}$). However, $A^T = \begin{pmatrix} 1 & j \\ i & k \end{pmatrix}$ is *not* invertible, with $\mathrm{rk}\,(A^T) = 1$ since $(i, k) = i \cdot (1, j)$.

Before coming to results on the rank of symmetric and skew-symmetric matrices over commutative rings and division rings, we must first understand reasonably well what happens over fields (or integral domains). If A is a *symmetric* matrix of rank r *over a field,* it is well known in linear algebra that A has a nonsingular $r \times r$ principal submatrix; see, e.g. [S: Th. 5.9, p. 118]. The same fact also holds for *skew-symmetric* matrices, and was given as an exercise in [S: Prob. 3, p. 119]. We shall refer to these facts (over a field) as the classical *Principal Minor Theorem* (PMT). In the following, we would like to investigate possible extensions of this theorem to the setting of commutative rings and division rings. Our approach is inspired by the work [HS] of Horn and Sergeichuk, which implied a generalized version of PMT for fields with involutions (via a condition on null spaces — see (HS) below). Before we introduce involutions, let us first define the following useful matrix-theoretic notion.

Definition 2.6. For any ring K, we say that a matrix $A \in \mathbb{M}_n(K)$ is *quasi-symmetric* if an arbitrary set of rows of A is left linearly independent iff the set of the corresponding columns of A is right linearly independent.

Examples 2.7. (1) Let R be a commutative ring. Clearly, *a symmetric or skew-symmetric matrix* $A \in \mathbb{M}_n(R)$ *is quasi-symmetric,* since a column of A is, up to a sign, the transpose of the corresponding row, and left and right linear independence for n-tuples are equivalent (as long as R is commutative). However, even over a division ring K, there are many quasi-symmetric matrices that are neither symmetric nor skew-symmetric. For instance, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(K)$ with $a,\, d \neq 0$ is always quasi-symmetric, but need not be symmetric or skew-symmetric (as b and c are completely arbitrary). The same statement also holds with $b,\, c \neq 0$ instead.

(2) Following Horn and Sergeichuk, we may construct the following large class of examples of quasi-symmetric matrices. Let K be a ring with an involution $x \mapsto \overline{x}$. (For instance, if K happens to be a commutative ring, we could take this to be the identity map.) For a matrix $A = (a_{ij}) \in \mathbb{M}_{m,n}(K)$, let A^* denote

[1] This property does not hold in general for matrices over commutative rings. See (3.13) and the ensuing discussions.

the transpose of $\overline{A} := (\overline{a_{ij}})$. The point about "bar" being an involution is that the identity $(AB)^* = B^*A^*$ holds (while in general $(AB)^T = B^TA^T$ does not) whenever the multiplication AB is defined. It is also easy to check that, if K is a division ring (or a commutative ring) with involution, then $\mathrm{rk}\,(A) = \mathrm{rk}\,(A^*)$. (In particular, $A \in \mathrm{GL}_n(K)$ iff $A^* \in \mathrm{GL}_n(K)$.) In the case where $n = m$, we'll say that A *satisfies the condition* (HS) if

(HS) $\qquad \forall$ *row vector* $u \in K^n: \ A^* \cdot u^T = 0 \iff A \cdot u^T = 0.$

Clearly, symmetric and skew-symmetric matrices satisfy (HS) over a commutative ring K if "bar" is the identity, and so do Hermitian and skew-Hermitian matrices over the complex field if "bar" is the complex conjugation. The following shows that (HS) is a stronger notion than quasi-symmetry.

Proposition 2.8. *In the setting above* (*where* $(K, \,^{-})$ *is a ring with involution*), *any matrix* $A \in \mathbb{M}_n(K)$ *satisfying* (HS) *is quasi-symmetric.*

Proof. Let $\alpha_1, \dots, \alpha_n$ (resp. $\beta_1, \dots, \beta_n$) be the rows (resp. columns) of A. Suppose a certain set of rows of A is left linearly dependent. For convenience, we may assume that these rows are $\alpha_1, \dots, \alpha_r$. Then there is an equation $u \cdot A = 0$ where $u = (x_1, \dots, x_r, 0, 0, \dots)$, with $x_i \in K$ not all zero. Taking *, we get $A^* \cdot (\overline{x}_1, \dots, \overline{x}_r, 0, 0, \dots)^T = 0$, and hence by the implication "$\Rightarrow$" in (HS), $A \cdot (\overline{x}_1, \dots, \overline{x}_r, 0, 0, \dots)^T = 0$. Since the $\overline{x}_i$'s are not all zero, the columns $\beta_1, \dots, \beta_r$ are right linearly dependent. Conversely, if these r columns are right linearly dependent to begin with, a similar argument using "$\Leftarrow$" in (HS) shows the left linear dependence of $\alpha_1, \dots, \alpha_r$. $\square$

We are now in a position to generalize the classical PMT to the setting of division rings. The following result (2.9) is called the *Strong Principal Minor Theorem* in that, even in the case of fields, its statement gives more specific information than the PMT for symmetric and skew-symmetric matrices quoted earlier from [S] (or the one for matrices satisfying the condition (HS) in [HS]).

Strong PMT 2.9. *Let* $A \in \mathbb{M}_n(K)$ *be a quasi-symmetric matrix of rank* r *over a division ring* K. *Suppose a given* $r \times r$ *submatrix* B *of* A *is nonsingular. Then so is the principal* $r \times r$ *submatrix of* A *whose rows are collinear with those of* B. *Equivalently,* r *rows of* A *are left linearly independent iff the principal submatrix of* A *formed from these* r *rows is nonsingular. These conclusions apply, in particular, to any matrix* A *satisfying* (HS) *over a division ring with involution.*

Proof. After a reindexing of the rows and columns, we may assume that B is a submatrix of the matrix A' consisting of the first r rows of A. Then the first r rows of A are left linearly independent, and so the first r columns of A are right linearly independent. Since $r = \mathrm{rk}\,(A)$, the remaining columns of A must be right linear combinations of the first r, and therefore the same holds for A'. Since A' also has rank r, its first r columns must then be right linearly independent. This means that the $r \times r$ northwest corner of A is nonsingular, as desired. The last statement in the theorem follows from Prop. 2.8. $\square$

The theorem above *does not* hold over commutative rings. In fact, it fails already over direct products of fields, as we'll see from examples in §3 below.

Corollary 2.10. *Let $A \in \mathbb{M}_n(K)$ be a quasi-symmetric matrix over a division ring K. Let A_0 be the submatrix of A consisting of the first p rows of A, and let A_1 be the $p \times p$ principal submatrix of A_0. The following statements are equivalent:*

(1) *All $p \times p$ submatrices of A_0 are singular.*
(2) *All principal submatrices of A containing A_1 are singular.*

If all principal submatrices of A containing A_1 of size $\leq 2p$ are quasi-symmetric, (1) *and* (2) *are also equivalent to:*

(3) *All principal submatrices of A containing A_1 of size $\leq 2p$ are singular.*

Proof. (1) $\Rightarrow$ (2). Assume some principal submatrix of A containing A_1 is nonsingular. Then the rows of this matrix are left linearly independent, and hence so are those of A_0. The latter means that A_0 has rank p, and so some $p \times p$ submatrix of A_0 is nonsingular.

(2) $\Rightarrow$ (1). Suppose some $p \times p$ submatrix of A_0 is nonsingular. Then the rows of A_0 are left linearly independent, so $p \leq r := \mathrm{rk}\,(A)$. We can thus add $r - p$ suitable rows to A_0 to get r left linearly independent rows. By (2.9), the principal $r \times r$ submatrix supported by these rows is nonsingular — and this principal submatrix contains A_1.

Of course, (2) $\Rightarrow$ (3) is a tautology. With the additional assumption that all principal submatrices of A containing A_1 of size $\leq 2p$ are quasi-symmetric, we finish by proving the following.

(3) $\Rightarrow$ (1). Suppose some $p \times p$ submatrix B of A_0 is nonsingular. Consider the smallest principal submatrix A' of A that contains both A_1 and B. Then A' has size $\leq 2p$ and is (by assumption) quasi-symmetric. Applying (2) $\Rightarrow$ (1) to A', we conclude that A' has a principal submatrix (necessarily of size $\leq 2p$) containing A_1 that is nonsingular. $\square$

Returning to alternating matrices, we retrieve the following classical result.

Corollary 2.11. [Fr] *If $A \in \mathbb{A}_n(F)$ where F is a field, then $r := \mathrm{rk}\,(A)$ is even.*

Proof. Since A is skew-symmetric and therefore quasi-symmetric (F being commutative), (2.9) applies. Thus, A has a nonsingular $r \times r$ principal submatrix, say C. (Of course, all we need is the *classical* Principal Minor Theorem for skew-symmetric matrices.) Then $C \in \mathrm{GL}_r(F) \cap \mathbb{A}_r(F)$ implies that r is even. $\square$

Remark 2.12. (1) As was pointed out by Professor C. K. Li, in the case of characteristic $\neq 2$, the existence of a nonzero $r \times r$ principal minor in an alternating matrix A of rank r over a field can also be deduced from Albert's result [Al: Th. 4, p. 391] that A is congruent to the orthogonal sum of a hyperbolic matrix and a zero matrix. (A *hyperbolic matrix* is an orthogonal sum of copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.) On the other hand, the fact that $\mathrm{rk}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1$ taken over a field of characteristic 2 shows that (2.11) is not true (in general) for symmetric or skew-symmetric matrices.

(2) Unfortunately, there is no analogue of (2.11) for division rings (or division rings with involutions). Let K be the division ring of real quaternions with the

canonical involution (taking i, j, k to $-i, -j, -k$). The alternating matrix $A = \begin{pmatrix} 0 & i & j \\ -i & 0 & k \\ -j & -k & 0 \end{pmatrix}$ was shown to be *nonsingular* in [LS: (4.9)], so $\mathrm{rk}\,(A) = 3$. Here, A has all the desirable matricial properties: we have $A^* = A$, so A is "Hermitian". In particular, A satisfies (HS) and is quasi-symmetric; but it has *odd* rank!

3. Determinantal Ideals and PMT Over Commutative Rings

The main goal of this section is to generalize some of the results in §2, to the extent possible, to the case of matrices over commutative rings. The generalization of PMT is somewhat tricky since the verbatim statement of this theorem turns out to be false for commutative rings in general. Reserving this for the second half of §3, we first take up here the easier generalization of (2.11).

Theorem 3.1. *Let $A \in \mathbb{A}_n(R)$ where R is a nonzero commutative ring.*

(1) *For any $k \geq 1$, $\mathrm{rad}\,(\mathcal{D}_{2k-1}(A)) = \mathrm{rad}\,(\mathcal{D}_{2k}(A))$.*
(2) $\mathrm{rk}\,(A)$ *is even.*
(3) *If R is a reduced ring and $k \geq 1$, $\mathrm{ann}\,(\mathcal{D}_{2k-1}(A)) = \mathrm{ann}\,(\mathcal{D}_{2k}(A))$.*

Proof. (1) It suffices to show that any prime $\mathfrak{p} \lhd R$ containing $\mathcal{D}_{2k}(A)$ also contains $\mathcal{D}_{2k-1}(A)$. Let "bar" denote the projection map from R to the integral domain $\overline{R} = R/\mathfrak{p}$. By assumption, $\mathcal{D}_{2k}(\overline{A}) = 0$, so $\mathrm{rk}\,(\overline{A}) \leq 2k-1$. Applying (2.11) over the quotient field of $\overline{R}$, we have $\mathrm{rk}\,(\overline{A}) \leq 2k-2$, and so $\mathcal{D}_{2k-1}(\overline{A}) = 0$. This shows that $\mathcal{D}_{2k-1}(A) \subseteq \mathfrak{p}$, as desired.

(2) Since $\mathcal{D}_{2k-1}(A)$ is f.g., (1) implies $\mathcal{D}_{2k-1}(A)^t \subseteq \mathcal{D}_{2k}(A) \subseteq \mathcal{D}_{2k-1}(A)$ for some t. By (2.1), $\mathcal{D}_{2k}(A)$ is dense in R iff $\mathcal{D}_{2k-1}(A)$ is. This clearly implies (2).

(3) Now assume R is reduced. For any $r \in R$, we need to show that

$$r \cdot \mathcal{D}_{2k}(A) = 0 \Longrightarrow r \cdot \mathcal{D}_{2k-1}(A) = 0. \tag{3.2}$$

This is obvious over integral domains (since $\mathrm{rk}\,(A)$ is even). Consider any prime ideal $\mathfrak{p} \lhd R$. Using the notations in (1), we have $\overline{r} \cdot \mathcal{D}_{2k}(\overline{A}) = 0$, so $\overline{r} \cdot \mathcal{D}_{2k-1}(\overline{A}) = 0$ by the domain case. Thus, $r \cdot \mathcal{D}_{2k-1}(A) \subseteq \bigcap_{\mathfrak{p}} \mathfrak{p} = \mathrm{Nil}\,(R) = 0$. $\square$

Remark 3.3. (3) above need not hold if R is not reduced. For instance, if $R = \mathbb{Q}\,[x]$ with the relation $x^2 = 0$, then for $A = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$, we have $\mathrm{ann}\,(\mathcal{D}_2(A)) = \mathrm{ann}\,(0) = R$, while $\mathrm{ann}\,(\mathcal{D}_1(A)) = \mathrm{ann}\,(xR) = xR$. Here, $\mathrm{rk}\,(A) = 0$.

Since the conclusion (3.1)(1) seems new, it is tempting to try also to compute (for given n and k) the smallest integer t such that $\mathcal{D}_{2k-1}(A)^t \subseteq \mathcal{D}_{2k}(A)$, say for the generic alternating matrix A. So far, we are able to do this only in the three special cases $k = 1$, $2k = n$, and $2k = n+1$. The last case is trivial, since $2k = n+1$ implies that $\mathcal{D}_{2k-1}(A) = \mathcal{D}_n(A) = 0$ so we can take $t = 1$. As for the first case $k = 1$, we have the following result.

Theorem 3.4. *Let $A = (a_{ij}) \in \mathbb{A}_n(R)$ over a commutative ring R, and let $t = [n/2] + 1$. Then* (1) $\mathcal{D}_1(A)^t \subseteq \mathcal{D}_2(A)$. (2) *If $1/2 \in R$, then $\mathcal{D}_1(A)^2 \subseteq \mathcal{D}_2(A)$.* (3) *If R and A are arbitrary, the exponent t in* (1) *is already the best possible.*

Proof. (1) Consider a product $a_{i_1j_1}a_{i_2j_2}\dots a_{i_tj_t} \in \mathcal{D}_1(A)^t$. Since $2t > n$, two indices must be equal. If a_{ii} occurs, we are done (since $a_{ii} = 0$). Otherwise there is a factor $a_{ij}a_{ki}$ (using $a_{pq} = -a_{qp}$), and we have

$$a_{ij}a_{ki} = a_{ij}a_{ki} - a_{ii}a_{kj} \in \mathcal{D}_2(A). \tag{3.5}$$

(2) Assuming $1/2 \in R$, we have to show that $a_{ij}a_{k\ell} \in \mathcal{D}_2(A)$ (for all indices). If the indices are not distinct, we may assume either $i = j$ or $i = \ell$ (using $a_{pq} = -a_{qp}$), and we are done again by (3.5). If the indices are distinct, $\mathcal{D}_2(A)$ contains

$$a_{ij}a_{k\ell} - a_{i\ell}a_{kj}, \quad -a_{ij}a_{\ell k} + a_{ik}a_{\ell j}, \quad a_{ik}a_{j\ell} - a_{i\ell}a_{jk}, \tag{3.6}$$

and these add up to $2a_{ij}a_{k\ell}$, proving (2).

(3) Consider the ring $R = \mathbb{F}_2\,[x_{ij}]$ (where $i, j \in [1, n]$) with relations dictated by the condition that $A := (x_{ij})$ is alternating, with $\mathcal{D}_2(A) = 0$; that is,

$$x_{ii} = 0, \quad x_{ji} = -x_{ij} = x_{ij}, \quad \text{and} \quad x_{ij}x_{k\ell} = x_{i\ell}x_{kj}$$

for all i, j, k, ℓ. We wish to show that the index of nilpotency of the maximal ideal $\sum x_{ij}\, R \lhd R$ is $[n/2] + 1$. The key observation here is that R is isomorphic to the subring of evenly graded terms of the $\mathbb{Z}_2$-graded ring $\mathbb{F}_2\,[z_1, ..., z_n]/(z_1^2, ..., z_n^2)$, via the map taking each x_{ij} to z_iz_j. That there is such a surjective homomorphism is obvious. For injectivity, note that both rings are 2^{n-1}-dimensional as $\mathbb{F}_2$-vector spaces. (This needs some calculations; cf. also the proof of (6.6).) Now a nonzero term of largest degree in the ring $\mathbb{F}_2\ [z_1, ..., z_n]/(z_1^2, ..., z_n^2)$ is of degree n. Thus, that of largest degree in the x_{ij} 's is $[n/2]$, which is what we wanted to see. □

As for the case of $\mathcal{D}_{n-1}(A)$, the study of this ideal and its relationship to $\mathcal{D}_n(A) = \det(A) \cdot R$ is a part of classical determinant theory (see, e.g. [Ca], [He]). For the reader's convenience, we include a short exposition on a couple of the key results below. First some more notations: for $A \in \mathbb{A}_n(R)$ and for any i, let $\mathcal{D}_i^*(A)$ be the ideal generated by the $i \times i$ *principal* minors of A, and let $\mathcal{P}_i(A)$ be the ideal generated by the Pfaffians of all $i \times i$ principal submatrices of A. These ideals are of interest only for i even, since they are zero when i is odd. Note that the classical Pfaffian expansion formula along a row [Mu: Art. 409] implies that

$$R := \mathcal{P}_0(R) \supseteq \mathcal{P}_2(A) \supseteq \mathcal{P}_4(A) \supseteq \cdots, \tag{3.7}$$

although this chain relation in general need not hold for the $\mathcal{D}_{2k}^*(A)$'s.

Theorem 3.8. *Let $A = (a_{ij}) \in \mathbb{A}_n(R)$, where R is any ring.*

(1) *If n is even, then $\mathcal{D}_{n-1}(A) = \mathrm{Pf}\,(A) \cdot \mathcal{P}_{n-2}(A) = \mathcal{P}_n(A) \cdot \mathcal{P}_{n-2}(A)$.*
In particular, $\mathcal{D}_{n-1}(A)^2 = \det(A) \cdot \mathcal{P}_{n-2}(A)^2 \subseteq \mathcal{D}_n(A)$.

(2) *If n is odd, then $\mathcal{D}_{n-1}(A) = \mathcal{P}_{n-1}(A)^2$.*

Proof. Both of these interesting facts are based on a determinantal identity from [Mu] relating the various "big minors" of a square matrix. In general, for any matrix A, let $A_{r,s}$ denote the determinant of the submatrix of A obtained by deleting the r-th row and the s-th column, and let $A_{rr',ss'}$ denote the determinant of the submatrix of A obtained by deleting the r-th and r'-th rows and the s-th and s'-th columns. For $r \neq s$, there is a basic quadratic minor relation:

$$A_{r,r}\, A_{s,s} - A_{r,s}\, A_{s,r} = (\det A)\, A_{rs,rs}. \tag{3.9}$$

This identity — for any A — can be found in [Mu: Art. 175]. Here, we apply it to the case where $A \in \mathbb{A}_n(R)$.

(1) *Assume n is even.* Then $A_{r,r} = A_{s,s} = 0$, and it is easy to see that $A_{s,r} = -A_{r,s}$. Therefore, (3.9) simplifies to

$$(A_{r,s})^2 = (\det A)\, A_{rs,rs} = \mathrm{Pf}\,(A)^2 A_{rs,rs}. \tag{3.9$'$}$$

To prove (1), we may assume that the entries a_{ij} $(i < j)$ are independent (commuting) indeterminates, and that R is the polynomial ring (over $\mathbb{Z}$) in these a_{ij}'s. (If (1) holds in this case, then it holds in general by specialization.) But now R is a domain, so taking square roots in (3.9)$'$ yields

$$A_{r,s} = \pm\,\mathrm{Pf}\,(A) \cdot \mathrm{Pf}\,\{r, s\}', \tag{3.10}$$

where, to conform with notations to be used later in §4, $\mathrm{Pf}\,\{r, s\}'$ denotes the Pfaffian of the submatrix of A obtained by deleting its r-th and s-th rows and r-th and s-th columns. This proves the first part of (1), from which the second part follows by squaring. (An interesting consequence of the first part of (1) is that, for n even, A has a nonsingular $(n-1)\times(n-1)$ nonprincipal submatrix iff A is nonsingular and A has a nonsingular $(n-2)\times(n-2)$ principal submatrix.)

(2) *Assume now n is odd.* This implies that $A_{rs,rs} = 0$, and $A_{s,r} = A_{r,s}$. Thus, (3.9) yields $(A_{r,s})^2 = A_{r,r}A_{s,s}$. By the same trick of going over to the generic case, we see in general that

$$A_{r,s} = \pm\,\mathrm{Pf}\,\{r\}' \cdot \mathrm{Pf}\,\{s\}', \tag{3.11}$$

where again $\mathrm{Pf}\,\{i\}'$ denotes the Pfaffian of the submatrix of A obtained by deleting its i-th row and i-th column. Since $\mathcal{D}_{n-1}(A)$ is generated by the $A_{r,s}$'s and $\mathcal{P}_{n-1}(A)$ is generated by the $\mathrm{Pf}\,\{i\}'$'s, it follows that $\mathcal{D}_{n-1}(A) = \mathcal{P}_{n-1}(A)^2$. For some more discussions on this nice classical formula, see [BS]. □

Next, we give a variation on the theme of (3.1)(1) by replacing the $\mathcal{D}_i(A)$ there with the ideal $\mathcal{D}_i^*(A)$ generated by the $i \times i$ principal minors.

Theorem 3.12. *For $A \in \mathbb{A}_n(R)$ and any $i \geq 0$, we have the following.*

(1) $\mathcal{P}_i(A)^{N+1} \subseteq \mathcal{D}_i^*(A) \subseteq \mathcal{P}_i(A)^2$ *where* $N = \binom{n}{i}$.
(2) $\mathrm{rad}\,(\mathcal{D}_i^*(A)) = \mathrm{rad}\,(\mathcal{P}_i(A))$.
(3) $\mathrm{rad}\,(\mathcal{D}_i^*(A)) = \mathrm{rad}\,(\mathcal{D}_i(A))$ *for i even.*
(4) $\mathrm{rk}\,(A) = \max\{\, j : \mathcal{D}_j^*(A)$ *is dense*$\,\} = \max\{\, j : \mathcal{P}_j(A)$ *is dense*$\,\}$.

Proof. (1) For any $i \times i$ principal submatrix C of A, we have $\det(C) = \mathrm{Pf}\,(C)^2$, so $\mathcal{D}_i^*(A) \subseteq \mathcal{P}_i(A)^2$. On the other hand, if $C_1, \ldots, C_{N+1}$ are $i \times i$ principal submatrices, then two of them must be the same, so $\mathrm{Pf}\,(C_1)\cdots\mathrm{Pf}\,(C_{N+1})$ has a factor $\mathrm{Pf}\,(C_k)^2 = \det(C_k) \in \mathcal{D}_i^*(A)$. This proves $\mathcal{P}_i(A)^{N+1} \subseteq \mathcal{D}_i^*(A)$.

Clearly, (1) ⇒ (2), and (2) + (3) ⇒ (4) follows from (2.1). To prove (3) (with i even), it suffices to show that every prime $\mathfrak{p} \supseteq \mathcal{D}_i^*(A)$ contains $\mathcal{D}_i(A)$. Now all $i \times i$ principal minors of A vanish in $\overline{R} = R/\mathfrak{p}$. If $\mathfrak{p} \not\supseteq \mathcal{D}_i(A)$, some $i \times i$ minor of $\overline{A}$ doesn't vanish in $\overline{R}$. Thus, $r := \mathrm{rk}\,(\overline{A}) \geq i$. By (2.9), some $r \times r$ principal minor of $\overline{A}$ does not vanish in $\overline{R}$. Since r (as well as i) is even, (3.7) implies that some $i \times i$ *principal* minor of $\overline{A}$ doesn't vanish in $\overline{R}$ — a contradiction. □

To study possible extensions of PMT (the classical Principal Minor Theorem) to commutative rings, we must first address the following basic question:

(3.13) *If a matrix A over a ring R has rank r (so the ideal $\mathcal{D}_r(A)$ is dense), when can we say that A has an $r \times r$ nonsingular submatrix?*

A quick informal answer is "not always yes". Indeed, over the semisimple ring $R = \mathbb{Z}/6\,\mathbb{Z} \cong \mathbb{F}_2 \times \mathbb{F}_3$, the matrix $(2,\, 3)$ has rank 1, but no 1×1 submatrix is nonsingular. The main property of the ring R responsible for this phenomenon is that, while $2,\, 3 \in \mathcal{Z}(R)$, they are killed simultaneously only by 0. This idea can, in fact, be generalized to yield the following answer for Question (3.13).

Theorem 3.14. *For a given ring $R \neq 0$, the following statements are equivalent*:

(1) *Every rectangular matrix A over R has a nonsingular submatrix of size* $\mathrm{rk}\,(A)$.
(2) *Every alternating matrix A over R has a nonsingular submatrix of size* $\mathrm{rk}\,(A)$.
(3) *R has the following property*:

$(*)$ *For every finite subset $X \subseteq \mathcal{Z}(R)$,* $\mathrm{ann}\,(X) \neq 0$.

Proof. (1) $\Rightarrow$ (2) is a tautology. To prove (2) $\Rightarrow$ (3), assume (3) is false. Then there exist $x_1, \ldots, x_n \in \mathcal{Z}(R)$ for which $tx_i = 0\ (\forall\, i) \Rightarrow t = 0$. Let $A \in A_{n+1}(R)$ be such that its first row is $(0, x_1, \ldots, x_n)$, its first column is $(0, -x_1, \ldots, -x_n)^T$, and all of its other entries are 0. Then $\mathcal{D}_1(A)$ is dense as it contains $x_1, \ldots, x_n$. On the other hand, clearly $\mathcal{D}_k(A) = 0$ for $k \geq 3$. Since $\mathrm{rk}\,(A)$ is even (by (3.1)(2)), it must be 2. By quick inspection, however, a 2×2 minor of A is zero or $x_j x_k$, which are all in $\mathcal{Z}(R)$. Thus, (2) is false for A.

Finally, to check (3) $\Rightarrow$ (1), we assume (3). For any rectangular matrix A of rank r, let X be the finite set of all $r \times r$ minors of A. If $X \subseteq \mathcal{Z}(R)$, then by $(*)$, $a \cdot X = 0$ for some $a \in R \setminus \{0\}$. But then $a \cdot \mathcal{D}_r(A) = 0$ contradicts the denseness of $\mathcal{D}_r(A)$. Therefore, *some* $x \in X$ must be a non 0-divisor, which proves (1). $\square$

Note that the property $(*)$ implies that $\mathcal{Z}(R)$ is a prime ideal of R. Indeed, if $x,\, y \in \mathcal{Z}(R)$, we have by $(*)$ $ax = ay = 0$ for some $a \in R \setminus \{0\}$, so $a(x+y) = 0$ and hence $x + y \in \mathcal{Z}(R)$. This shows that $\mathcal{Z}(R)$ is an ideal, and the fact that $R \setminus \mathcal{Z}(R)$ is the *multiplicatively closed* set of all non 0-divisors of R shows that $\mathcal{Z}(R)$ is a prime ideal. Using this remark in conjunction with (3.14), we'll prove the following rank theorem over the class of rings satisfying $(*)$. Part (3) below is the strong form of the Principal Minor Theorem for symmetric and skew-symmetric matrices over this class of rings.

Theorem 3.15. *Let $R \neq 0$ be a ring with the property $(*)$. For every rectangular matrix A of rank r over R, we have following.*

(1) *r is the largest integer t for which A has an $t \times t$ nonsingular submatrix.*
(2) *r is also the largest integer s for which A has s linearly independent rows.*
(3) (Strong PMT) *Assume that $A^T = \pm A$ (so in particular A is a square matrix). If a given $r \times r$ submatrix B of A is nonsingular, then so is the principal $r \times r$ submatrix of A whose rows are collinear with those of B.*

Proof. We first note that, *without any assumptions,* $r \geq s \geq t$. First, by McCoy's Rank Theorem, s linearly independent rows of A define a submatrix S of rank s.

Thus, we have $r \geq \mathrm{rk}\,(S) = s$. On the other hand, take any $t \times t$ nonsingular submatrix T of A. The t rows of T are linearly independent, and hence so are the t rows of A containing them — showing $s \geq t$.

Now assume R satisfies $(*)$. By (3.14), A has a nonsingular $r \times r$ submatrix, so $t \geq r$. We have thus $r = s = t$, which proves (1) and (2).

For (3), we use our earlier remark that $\mathfrak{p} := \mathcal{Z}(R)$ is a prime ideal. This shows that $\overline{R} = R/\mathfrak{p}$ is an integral domain, and that a square matrix M over R is nonsingular iff its image $\overline{M}$ is nonsingular over the quotient field K of $\overline{R}$. Therefore, the number t in (1) is just the rank of the matrix $\overline{A}$ over K; that is, the rank of any matrix A is unchanged when we pass from R to K. With this observation, it is clear that the Strong PMT over K (from (2.9)) implies the Strong PMT over R. $\square$

One interesting class of rings satisfying $(*)$ (in generalization of the obvious class of integral domains) is as follows.

Example 3.16. Let $R \neq 0$ be any ring in which $\mathcal{Z}(R) = \mathrm{Nil}\,(R)$. (This is equivalent to saying that (0) is a primary ideal in R.) *Then R has the property* $(*)$. To see this, consider any finite set of elements $x_1, \ldots, x_n \in \mathcal{Z}(R)$. Since each x_i is nilpotent, we can choose $a := x_1^{i_1} \cdots x_n^{i_n} \neq 0$ with $i_1 + \cdots + i_n$ *maximal*. Then $ax_i = 0$ for all i, as desired. For instance, R can be any 0-dimensional local ring. More generally, we can take $R = S/\mathfrak{q}$ where $\mathfrak{q}$ is a primary ideal in any ring S. Here, $\mathcal{Z}(R) = \mathfrak{p}/\mathfrak{q}$, where $\mathfrak{p} = \mathrm{rad}\,(\mathfrak{q})$ is the associated prime of $\mathfrak{q}$.

We'll come back at the end of this section to say a few more things about $(*)$. Here, we offer an example to show that the strong form of PMT as stated in (3.15)(3) is *not* true — even for alternating matrices — over a semisimple ring.

Example 3.17. Over the ring $R = \mathbb{Z}/6\,\mathbb{Z}$ again, let

$$A = \begin{pmatrix} 0 & 3 & 2 & 3 \\ -3 & 0 & 3 & 2 \\ -2 & -3 & 0 & 3 \\ -3 & -2 & -3 & 0 \end{pmatrix} \in \mathbb{A}_4(R). \tag{3.18}$$

An easy computation shows that $\mathrm{rk}\,(A) = 2$. (Here, $\mathrm{Pf}\,(A) = 2 \in \mathcal{Z}(R)$, and $r = s = t = 2$ in the notations of (3.15).) The first two rows contain the nonsingular (even invertible) submatrix $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$; however, the principal submatrix supported by these two rows is singular. In fact, *all 2×2 principal submatrices of A are singular* since their Pfaffians are the above-diagonal entries of A, which are all 0-divisors. This shows that even the *classical* form of PMT cannot hold over a direct product of two fields. In the example above, it is of interest to note that, over the factor ring $R/2R \cong \mathbb{F}_2$, the image of A has rank 2, while over $R/3R \cong \mathbb{F}_3$, the image of A has rank 4.

In spite of counterexamples of the above nature, there does exist one special case in which the strong form of PMT happens to hold without *any* conditions on the ring R. This special case is made possible by the quadratic minor relation (3.9) used to prove Theorem 3.8.

Proposition 3.19. *Let* $A \in \mathbb{M}_n(R)$ *be such that* $A^T = \pm A$ *and* $\det(A) = 0$. *If a given* $(n-1)\times(n-1)$ *submatrix* B *of* A *is nonsingular (resp. invertible), then so is the principal* $(n-1)\times(n-1)$ *submatrix of* A *whose rows are collinear with those of* B.

Proof. Note that $\det(A) = 0$ and the nonsingularity of B imply that $\operatorname{rk}(A) = n-1$, though this fact is not essential to the proof below. We may assume that B is a *nonprincipal* submatrix (for otherwise there is nothing to prove). Using the notations in the proof of (3.8), we have $\det(B) = A_{r,s}$ for some $r \neq s$. Since $\det(A) = 0$, (3.9) implies that $A_{r,r}A_{s,s} = A_{r,s}A_{s,r}$. From $A^T = \pm A$, we see easily that $A_{s,r} = \pm A_{r,s}$, so $A_{r,r}A_{s,s} = \pm A_{r,s}^2$. Since $A_{r,s} \notin \mathcal{Z}(R)$ (resp. $A_{r,s}$ is a unit), the same holds for $A_{r,r}$. This means precisely that the principal $(n-1)\times(n-1)$ submatrix of A whose rows are collinear with those of B is nonsingular (resp. invertible). $\square$

Remark 3.20. Recall that, if $A \in \mathbb{A}_n(R)$ and n is odd, the condition $\det(A) = 0$ is automatic. Thus, for n odd, the conclusion of (3.19) applies to all $A \in \mathbb{A}_n(R)$. For n even, however, (3.19) is vacuous for $A \in \mathbb{A}_n(R)$, in view of (3.8)(1).

In deference to the interesting role played by $(*)$ in the two results (3.14) and (3.15), we close this section by making some supplementary remarks about this property. To take a broader view, consider the following related 0-divisor properties, where n denotes a (possibly infinite) cardinal number ≥ 2:

$(*)_n$ $\operatorname{ann}(X) \neq 0$ *for every subset* $X \subseteq \mathcal{Z}(R)$ *with* $|X| \leq n$; and
$(**)$ $\mathcal{Z}(R)$ *is an ideal in* R.

Of course, if $n \geq m \geq 2$ (as cardinals), we have $(*)_n \Rightarrow (*)_m \Rightarrow (**)$, and the original property $(*)$ in Theorem 3.14 is just the conjunction of the properties $(*)_n$ for all finite cardinals n.

In case $R \neq 0$ *is noetherian, all of the above properties are equivalent.* For, if $(**)$ holds, then $\operatorname{ann}(\mathcal{Z}(R)) \neq 0$ by [Ka: Th. 82], and so R has the property $(*)_n$ for all cardinals n. For general commutative rings, however, *the properties* $(*)_n$ *and* $(**)$ *are mutually distinct.* Since this fact may not be well known, we give a sketch of the necessary arguments below to show this distinctness.

Example 3.21. Let $(A, \mathfrak{m}_0)$ be a local domain, and let $T \neq 0$ be a A-module on which every element of $\mathfrak{m}_0$ acts as a 0-divisor. Let $R = A \oplus T$ be the *split-null extension* (a.k.a. the *trivial extension*) of A by T, which is a ring with the commutative multiplication

$$(a+t)(a'+t') = a\,a' + (a\,t' + a't) \quad \text{for all } a,\, a' \in A \text{ and } t,\, t' \in T, \tag{3.22}$$

dictated by the stipulation that $T^2 = 0$ in R. It is easy to check that R is a local ring with the unique maximal ideal $\mathfrak{m} := \mathfrak{m}_0 \oplus T$, and that $\mathfrak{m} = \mathcal{Z}(R)$. *The latter shows that* R *has the property* $(**)$. In the following, we'll further specialize this construction to get the examples we want.

(1) To show that $(*)_m \not\Rightarrow (*)_n$ where $n > m \geq 2$ are integers, take

$$A = \mathbb{Q}\,[x_1,\, \ldots,\, x_n\,]_{(x_1,\ldots,\,x_n)}, \tag{3.23}$$

with maximal ideal $\mathfrak{m}_0 = \sum_i Ax_i$, and let $T = \bigoplus A/\mathfrak{p}$ with $\mathfrak{p}$ ranging over all prime ideals of A except $\mathfrak{m}_0$. Since every $a \in \mathfrak{m}_0$ is contained in a height 1 prime (and $n \geq 2$), a acts as a 0-divisor on T. In the resulting split-null extension $R = A \oplus T$, we have $\operatorname{ann}_R\{x_1, \ldots, x_n\} = 0$, since $\{x_1, \ldots, x_n\}$ has zero annihilator in A and in $A/\mathfrak{p}$ for each prime $\mathfrak{p} \neq \mathfrak{m}_0$. Thus, R *does not satisfy* $(*)_n$. However, for any $\{y_1, \ldots, y_m\} \subseteq \mathcal{Z}(R) = \mathfrak{m}_0 \oplus T$ with $m < n$, a prime $\mathfrak{p}$ in A minimal over the A-coordinates of the y_i's cannot equal $\mathfrak{m}_0$, so $\operatorname{ann}_{A/\mathfrak{p}}\{y_1, \ldots, y_m\} \neq 0$. This checks that R *satisfies* $(*)_m$ *for all* $m < n$. The special case of this construction for $n = 2$ also shows that $(**) \not\Rightarrow (*)_2$.

(2) For a fixed *infinite* cardinal n, it is also easy to use the same construction in (1) to show that $(*)_n$ is not implied by $(*)$ or by $(*)_m$ for any cardinal $m < n$. The construction in (1) (for infinite n) yields a split-null extension R in which the set $\{x_i\}_{i \in I} \subseteq \mathcal{Z}(R)$ with $|I| = n$ has zero annihilator as before. Now consider any set $Y \subseteq \mathcal{Z}(R) = \mathfrak{m}_0 \oplus T$ with $|Y| = m < n$. Again, it suffices to work in the case where $Y \subseteq \mathfrak{m}_0$, and we may assume that the elements of Y are polynomials (with zero constant terms). Since each $y \in Y$ can be expressed in terms of a finite number of variables, there exists an indexing set $J \subseteq I$ with $|J| < n$ such that each $y \in Y$ is a polynomial in $\{x_i\}_{i \in J}$. Thus, $Y \subseteq \mathfrak{p} := \sum_{i \in J} Ax_i$. Since $\mathfrak{p}$ is prime and not equal to $\mathfrak{m}_0$, we see that $\operatorname{ann}_R(Y) \supseteq \operatorname{ann}_{A/\mathfrak{p}}(Y) \neq 0$. Thus, the ring R satisfies $(*)_m$ for every $m < n$ (and in particular $(*)$).

4. A Pfaffian Expansion Formula

In this section, we study the Pfaffians of an alternating matrix and its principal submatrices from an ideal-theoretic point of view (over a commutative ring R), and prove a certain expansion formula for the Pfaffian that generalizes the classical formula for its "row expansion". While the row expansion formula is quite well known (see, for instance, [Mu: Art. 409]), a modest search of the literature did not turn up the kind of more general expansion formula we want. Since the latter formula has a nice application to the study of the Pfaffian ideals (generated by various sub-Pfaffians of an alternating matrix), we record the formula in this section and present for it a self-contained combinatorial proof. Some applications of this formula are given in (4.4) and (8.1) below; see also Remark 5.6.

We first introduce the following notations. For a matrix $A = (a_{ij}) \in \mathbb{M}_n(R)$ and any subset $J \subseteq \{1, \ldots, n\}$, we write $J' := \{1, \ldots, n\} \setminus J$, and let $A[J]$ be the principal submatrix of A consisting of the entries a_{ij} for which $i, j \in J$. As we have noted before, if A is alternating, so are $A[J]$ and $A[J']$. *Throughout this section, we assume* $n = 2m$ *is even, and* $A \in \mathbb{A}_n(R)$. Recall that $\operatorname{Pf}(A)$ is a sum of terms of the form

(4.1) $$\pm\, a_{i_1,j_1} \cdots a_{i_m,j_m}, \ \text{ where } i_s < j_s \ (\forall\, s), \ \text{ and } i_1 < \cdots < i_m.$$

Here, $i_1, j_1, \ldots, i_m, j_m$ is a permutation of $1, \ldots, n$, and "$\pm$" is the sign of this permutation. For starters, $\operatorname{Pf}\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} = x$, and for $A \in \mathbb{A}_4(R)$ as in (7.3) below, $\operatorname{Pf}(A) = xw - yv + zu$.

To write down the Pfaffian Expansion Formula, we *fix* a nonempty subset $J_0 \subseteq \{1, \ldots, n\}$, say of cardinality p. For convenience, we'll take $J_0 = \{1, \ldots, p\}$.

We'll call a subset $J \subseteq \{1, \ldots, n\}$ *allowable* if $J \supseteq J_0$, $|J|$ is even, and $|J| \leq 2p$. For such a set J, we write $|J| = 2p - 2k$ (so $k \geq 0$ is uniquely determined by J), and we define $\varepsilon(J)$ to be the sign of the permutation that takes the elements of J and J' listed (separately) in increasing order back to the natural order $1, \ldots, n$. With these notations, we can now state the formula we want.

Pfaffian Expansion Theorem 4.2. *For $A = (a_{ij}) \in \mathbb{A}_n(R)$ with $n = 2m$ and $J_0 = \{1, \ldots, p\}$ $(p \leq n)$, we have*

$$\text{(4.3)} \qquad \operatorname{Pf}(A) = \sum_J (-1)^k \binom{m-p+k-1}{k} \varepsilon(J) \operatorname{Pf}(A[J]) \operatorname{Pf}(A[J']),$$

where the sum is taken over all allowable sets $J \subseteq \{1, \ldots, n\}$, and $k := (2p - |J|)/2$.

Let us first record the following easy consequence of (4.2). Note that, in the special case of fields, (2) below echoes the theme of the equivalence (2) $\Leftrightarrow$ (3) in Corollary 2.10.

Pfaffian Vanishing Theorem 4.4. *Keep the notations in* (4.2).

(1) *Let $\mathfrak{A}$ be the ideal of R generated by $\operatorname{Pf}(A[J])$ where J ranges over all allowable subsets of $\{1, \ldots, n\}$. Then $\operatorname{Pf}(A[J_1]) \in \mathfrak{A}$ for every $J_1 \supseteq J_0$.*

(2) *If $\operatorname{Pf}(A[J]) = 0$ for every allowable J, then $\operatorname{Pf}(A[J_1]) = 0$ for every $J_1 \supseteq J_0$.*

Proof. (2) is just the special case of (1) when $\mathfrak{A} = 0$. For (1), we may assume that $|J_1|$ is even (for otherwise $\operatorname{Pf}(A[J_1]) = 0$). We may also assume that $|J_1| > 2p$ (for otherwise J_1 is already allowable). The desired conclusion then follows by applying (4.3) with $A[J_1]$ replacing A, and noting that, after this replacement, there are simply fewer allowable subsets w.r.t. $J_0 \subseteq J_1$. (Of course, this conclusion is of interest only in the case $p < m$; it is a tautology otherwise.) $\square$

Before coming to the proof of (4.2), it will be useful to see what happens in a few special cases, as follows.

(A) *The simplest case is, of course, where $p = 1$.* Here, $J_0 = \{1\}$, and an allowable set J is just a doubleton $\{1, j\}$ $(1 < j \leq n)$. Since $k = (2p - |J|)/2 = 0$, and $\varepsilon(\{1, j\}) = (-1)^j$, (4.3) simplifies to

$$\text{(4.5)} \qquad \operatorname{Pf}(A) = \sum_{j=2}^{n} (-1)^j \operatorname{Pf}\{1, j\} \operatorname{Pf}\{1, j\}',$$

where we have further abbreviated $\operatorname{Pf}(A[J])$ into $\operatorname{Pf} J$ for every set J. We use this abbreviation mainly in (4.5) and the example (C) below where the set J is given explicitly as a collection of indices. (This notation should be harmless as long as the matrix A is fixed and understood.) Since $\operatorname{Pf}\{1, j\} = a_{1j}$, (4.5) is just the standard formula we've mentioned before for the Pfaffian expansion of $\operatorname{Pf}(A)$ along its first row.

(B) *Next, we consider the case where $p \geq m$.* Here, the allowable sets are *all* subsets $J \supseteq J_0$ of even cardinality (since we have automatically $|J| \leq 2m \leq 2p$). If $J \neq \{1, \ldots, n\}$, then $2p - 2k = |J| < 2m$ shows that $k > 0$ and $m - p + k - 1 \in$

$[0,\, k-1]$. In this case, $\binom{m-p+k-1}{k} = 0$, so J only contributes a zero term to the LHS of (4.3). If $J = \{1, \dots, n\}$, we have $k = p - m \geq 0$ and hence

$$(-1)^k \binom{m-p+k-1}{k} \varepsilon(J) = (-1)^k \binom{-1}{k} \cdot 1 = (-1)^k \frac{(-1)^k k!}{k!} = 1.$$

Thus, the LHS of (4.3) is down to the one term $\mathrm{Pf}\,(A)$, which checks (4.3) (albeit trivially).

(C) For a more nontrivial illustration of (4.3), we take $p = 2$ and $n = 6$. Here, $J_0 = \{1, 2\}$, and the allowable sets J are

$$\{1,2\}, \text{ and } \{1,2,3,4\},\ \{1,2,3,5\},\ \{1,2,3,6\},\ \{1,2,4,5\},\ \{1,2,4,6\},\ \{1,2,5,6\},$$

with $\varepsilon(J)$ given by 1 and $1, -1, 1, 1, -1, 1$, respectively. Since $k = 1$ when $|J\,| = 2$, and $k = 0$ when $|J\,| = 4$, (4.3) boils down to

$$\begin{aligned} \mathrm{Pf}\,(A) = &-\mathrm{Pf}\,\{12\}\,\mathrm{Pf}\,\{3456\} + \mathrm{Pf}\,\{1234\}\,\mathrm{Pf}\,\{56\} - \mathrm{Pf}\,\{1235\}\,\mathrm{Pf}\,\{46\} \\ &+ \mathrm{Pf}\,\{1236\}\,\mathrm{Pf}\,\{45\} + \mathrm{Pf}\,\{1245\}\,\mathrm{Pf}\,\{36\} \\ &- \mathrm{Pf}\,\{1246\}\,\mathrm{Pf}\,\{35\} + \mathrm{Pf}\,\{1256\}\,\mathrm{Pf}\,\{34\}, \end{aligned}$$

where, of course, we could have replaced all $\mathrm{Pf}\,\{ij\}$ by a_{ij}. To double-check this formula for correctness, consider, e.g. the term $a_{12}a_{35}a_{46}$, which occurs on the LHS with the sign -1 (since (123546) is an odd permutation). The same term occurs on the RHS with coefficient $1 + 0 - 1 + 0 + 0 - 1 + 0$, which is, happily, also -1.

Since each term in the product $\mathrm{Pf}\,(A\,[J\,])\,\mathrm{Pf}\,(A\,[J\,'\,])$ does appear as a term in $\mathrm{Pf}\,(A)$, we can now try to use the same coefficient-comparison method in the $n = 6$ case above to check the general validity of (4.3). Our job is to verify that each "monomial term" $a_{i_1,j_1} \cdots a_{i_m,j_m}$ as in (4.1) occurs with the same coefficient on the two sides of the Expansion Formula (4.3). To this end, the following initial observation will be very helpful.

Lemma 4.6. *In the notations of* (4.2)*, let* $J \subseteq \{1, \dots, n\}$ *be allowable. Then a monomial term* $a_{i_1,j_1} \cdots a_{i_m,j_m}$ *as above occurs as a term in* $\mathrm{Pf}\,(A\,[J\,])\,\mathrm{Pf}\,(A\,[J\,'\,])$ *iff* J *is a union of some of the doubletons* $\{i_1, j_1\}, \dots, \{i_m, j_m\}$*. In this case, its coefficients in* $\mathrm{Pf}\,(A\,[J\,])\,\mathrm{Pf}\,(A\,[J\,'\,])$ *and in* $\mathrm{Pf}\,(A)$ *differ multiplicatively by* $\varepsilon(J)$*.*

Proof. The first statement is clear by inspection. For the second statement, assume J (and therefore also $J\,'$) is a union of some of the pairs $\{i_1, j_1\}, \dots, \{i_m, j_m\}$. Suppose it takes r (resp. s) transpositions to restore the union of these pairs in J (resp. in $J\,'$) to the increasing order. With J and $J\,'$ listed separately in increasing order, suppose it takes t transpositions to restore $\{J,\, J\,'\}$ back to $1, \dots, n\}$. Then the term $a_{i_1,j_1} \cdots a_{i_m,j_m}$ has coefficient $(-1)^r(-1)^s$ in $\mathrm{Pf}\,(A\,[J\,])\,\mathrm{Pf}\,(A\,[J\,'\,])$, while it has coefficient $(-1)^{r+s+t}$ in $\mathrm{Pf}\,(A)$. These coefficients differ multiplicatively by $(-1)^t = \varepsilon(J)$, as claimed. □

Proof of Thm. 4.2. Since we have already checked (4.3) in the case $p \geq m$ (see Example (B) above), *we may assume in the following that* $p \leq m$. (We could have assumed $p < m$, but this is not really necessary for the argument we are going to present.) Consider a fixed monomial $a_{i_1,j_1} \cdots a_{i_m,j_m}$ as in the last paragraph. In view of (4.6), it suffices to check that

$$1 = \sum_J (-1)^k \binom{m-p+k-1}{k}, \tag{4.7}$$

where the sum is over all allowable J's such that the term $a_{i_1,j_1} \cdots a_{i_m,j_m}$ features in $\mathrm{Pf}\,(A\,[J\,])\,\mathrm{Pf}\,(A\,[J\,'\,])$. (Recall that $k = (2p - |J\,|)/2$.) Now suppose exactly t of the pairs $\{i_s, j_s\}$ intersect J_0 $(t \le p)$. Then the number of allowable J's for which $a_{i_1,j_1} \cdots a_{i_m,j_m}$ features in $\mathrm{Pf}\,(A\,[J\,])\,\mathrm{Pf}\,(A\,[J\,'\,])$ is $\binom{m-t}{p-k-t}$. This is because such J must contain the t pairs $\{i_s, j_s\}$ that intersect J_0, and of the $m-t$ pairs left over, it must have $p-k-t$ of them to be of the right cardinality. Thus, after converting the RHS of (4.7) to a sum over k, our proof is reduced to checking the following binomial identity:

$$1 = \sum_{k=0}^{p-t} (-1)^k \binom{m-p+k-1}{k} \binom{m-t}{p-k-t} \tag{4.8}$$

where $1 \le p \le m$, and $t \le p$. This binomial identity holds since the RHS may be interpreted as the coefficient of x^{p-t} in

$$(1+x)^{-(m-p)} \cdot (1+x)^{m-t} = (1+x)^{p-t},$$

upon noting that $(1+x)^{m-t} = \sum_{\ell=0}^{m-t} \binom{m-t}{\ell} x^\ell$, and

$$(1+x)^{-(m-p)} = \sum_{k=0}^{\infty} (-1)^k \binom{m-p+k-1}{k} x^k. \qquad \square$$

5. A Row-Independence Criterion

The first result in this section is Theorem 5.1 on the radicals of certain determinantal ideals associated with a symmetric or skew-symmetric matrix A over a commutative ring R. This theorem is essentially an extension of Corollary 2.10 to the ring case (for the aforementioned types of matrices); its proof consists of a standard reduction to the field case, for which (2.10) is applicable. This theorem leads directly to a criterion in (5.4) on the linear independence of a set of rows of A in terms of the density of an ideal generated by suitable principal minors of A.

Throughout this section, we will use the following fixed notations. For $A \in \mathbb{M}_n(R)$ and for a fixed integer $p \le n$, let A_0 be the submatrix of A consisting of its first p rows, and let A_1 be the $p \times p$ northwest corner of A. We also define three ideals $\mathfrak{C}$, $\mathfrak{B}$, $\mathfrak{B}' \lhd R$ as follows:

- $\mathfrak{C}$ is the ideal of R generated by the maximal minors of A_0 (that is, $\mathfrak{C} = \mathcal{D}_p(A_0)$).
- $\mathfrak{B}$ be the ideal of R generated by the determinants of all principal submatrices of A containing A_1 of size $\le 2p$.
- $\mathfrak{B}'$ is the ideal of R generated by the determinants of all principal submatrices of A containing A_1. (Of course, $\mathfrak{B}' \supseteq \mathfrak{B}$.)

Theorem 5.1. *Let $A \in \mathbb{M}_n(R)$ be such that $A^T = \pm A$. Then, in the above notations, $\mathfrak{B} \subseteq \mathfrak{B}' \subseteq \mathfrak{C}$ and $\mathrm{rad}\,(\mathfrak{B}) = \mathrm{rad}\,(\mathfrak{B}') = \mathrm{rad}\,(\mathfrak{C})$.*

Proof. The fact that $\mathfrak{B}' \subseteq \mathfrak{C}$ follows from the p-row Laplace expansion of the determinant [Mu: Art. 93]. In view of this, the theorem will follow if we can show that $\mathfrak{C} \subseteq \operatorname{rad}(\mathfrak{B})$; or equivalently,

(5.2) *If $\mathfrak{P} \lhd R$ is any prime ideal containing $\mathfrak{B}$, then $\mathfrak{C} \subseteq \mathfrak{P}$.*

To check this, let K be the quotient field of $R/\mathfrak{P}$. Over K, all principal submatrices of A containing A_1 of size $\leq 2p$ are singular. By (2.10), all $p \times p$ submatrices of A_0 must also be singular (over K).[2] This means that the maximal minors of A_0 are all in $\mathfrak{P}$; that is, $\mathfrak{C} \subseteq \mathfrak{P}$, as desired. □

According to (5.1), *every maximal minor of the matrix A_0 has a power lying in the ideal $\mathfrak{B}$*. However, the abstract proof of the theorem did not lend itself to finding such powers explicitly. We can only illustrate the constructive aspect of (5.1) in some special cases below.

Examples 5.3. The case $p = 1$ of (5.1) is easy to verify directly. Indeed, if $A = (a_{ij}) = \pm A^T$, then $\mathfrak{C} = \sum_i a_{1i} R$. Since the generators of $\mathfrak{B}$ are a_{11} and $a_{11}a_{ii} \mp a_{1i}^2$ (for $i > 1$), we have $\mathfrak{B} = a_{11}R + \sum_{i=2}^n a_{1i}^2 R \subseteq \mathfrak{C}$. It is clear that $\operatorname{rad}(\mathfrak{B}) = \operatorname{rad}(\mathfrak{C})$ — without any assumptions on the a_{ii}'s.

Next, consider a skew-symmetric $A = \begin{pmatrix} a & x & y \\ -x & b & z \\ -y & -z & c \end{pmatrix}$, and take $p = 2$. Let $\alpha := ab + x^2$. Since $\det(A) = by^2 + az^2 + c\alpha$ (by expansion along the last row), a set of generators for $\mathfrak{B}$ may be taken to be α and $\beta := by^2 + az^2$. For the 2×2 minor $\sigma := xz - by \in \mathfrak{C}$, we have

$$\sigma^2 = x^2z^2 - 2bxyz + b^2y^2 = (\alpha - ab)\, z^2 + b\,(\beta - az^2) = z^2\,\alpha + b\,\beta \in \mathfrak{B},$$

using $2b = 0$. Similarly, for the 2×2 minor $\tau := az + xy \in \mathfrak{C}$, we have

$$\tau^2 = y^2\,\alpha + a\,\beta \in \mathfrak{B} \quad \text{and} \quad \sigma\,\tau = yz\,\alpha + x\,\beta \in \mathfrak{B},$$

using $2a = 2b = 0$ (but not $2c = 0$). This checks directly that $\mathfrak{C}^2 \subseteq \mathfrak{B}$.

The calculations in the case of *symmetric* matrices are a bit more intriguing. Let $A = \begin{pmatrix} a & x & y \\ x & b & z \\ y & z & c \end{pmatrix}$ and $p = 2$. As before, we can generate $\mathfrak{B}$ by

$$\alpha := ab - x^2 \quad \text{and} \quad \beta := 2xyz - by^2 - az^2.$$

For the 2×2 minor $\sigma := xz - by \in \mathfrak{C}$, we have

$$\begin{aligned} \sigma^2 &= x^2z^2 - 2bxyz + b^2y^2 \\ &= (ab - \alpha)\, z^2 - (b\,\beta + b^2y^2 + abz^2) + b^2y^2 \\ &= -z^2\,\alpha - b\,\beta \in \mathfrak{B}, \end{aligned}$$

with no conditions needed on a, b, c. Similarly, for the 2×2 minor $\tau := az - xy \in \mathfrak{C}$, we have $\tau^2 = -y^2\,\alpha - a\,\beta \in \mathfrak{B}$, and $\sigma\,\tau = -yz\,\alpha - x\,\beta \in \mathfrak{B}$, so again $\mathfrak{C}^2 \subseteq \mathfrak{B}$.

[2] The crucial implication (3) ⇒ (1) is applicable over K since here all principal submatrices of A are symmetric or skew-symmetric — and hence quasi-symmetric.

The nature of the calculations above suggests also that Theorem 5.1 would depend on the assumption $A^T = \pm A$. The following trivial example shows, for instance, that this symmetry/skew-symmetry condition cannot be weakened to quasi-symmetry. For the quasi-symmetric matrix $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ over $R = \mathbb{Z}$ with $p = 1$, we have $\mathfrak{C} = R$ and $\mathfrak{B} = 2R$, and thus $\operatorname{rad}(\mathfrak{C}) = R \neq 2R = \operatorname{rad}(\mathfrak{B})$.

A main application of Theorem 5.1 is the following result on the rows of a symmetric or skew-symmetric matrix over a commutative ring R.

Row-Independence Criterion 5.4. *Let $A \in \mathbb{M}_n(R)$ be such that $A^T = \pm A$ (where $R \neq 0$), and let $p \leq n$. Then, in the above notations, the first p rows of A are linearly independent over R iff the ideal $\mathfrak{B}$ is dense in R. In particular, if $\mathfrak{B}$ is a nil ideal (e.g. $\mathfrak{B} = 0$), then the first p rows of A must be linearly dependent.*

Proof. By (5.1), $\operatorname{rad}(\mathfrak{B}) = \operatorname{rad}(\mathfrak{C})$. Since $\mathfrak{B}$ and $\mathfrak{C}$ are both finitely generated, there exist integers N, N' such that $\mathfrak{B}^N \subseteq \mathfrak{C}$ and $\mathfrak{C}^{N'} \subseteq \mathfrak{B}$. Using (2.1), we see that $\mathfrak{B}$ is dense iff $\mathfrak{C}$ is dense. Now by definition $\mathfrak{C} = \mathcal{D}_p(A_0)$, where A_0 is the matrix consisting of the first p rows of A. Thus, $\mathfrak{C}$ is dense iff $\operatorname{rk}(A_0) = p$. By (2.2), this is the case iff the p rows of A_0 are linearly independent over R. $\square$

In the special case of an *alternating* matrix $A \in \mathbb{A}_n(R)$, we can bring another ideal into play; namely, the ideal $\mathfrak{A}$ of R (see (4.4)(1)) generated by the sub-Pfaffians $\operatorname{Pf}(A[J])$ where J ranges over all "allowable" subsets of $\{1, \ldots, n\}$ (that is, $J \supseteq \{1, \ldots, p\}$, $|J|$ is even, and $\leq 2p$). In light of (5.4), the following result on these Pfaffian ideals of A is entirely to be expected.

Corollary 5.5. *Let $A \in \mathbb{A}_n(R)$, and $p \leq n$ as before. Then* $\operatorname{rad}(\mathfrak{A}) = \operatorname{rad}(\mathfrak{C})$, *and the first p rows of A are linearly independent over R iff $\mathfrak{A}$ is dense in R.*

Proof. Since $\det(A[J]) = \operatorname{Pf}(A[J])^2$, it follows that $\mathfrak{B} \subseteq \mathfrak{A}^2$, and $\mathfrak{A} \subseteq \operatorname{rad}(\mathfrak{B})$. (Recall that odd sized principal Pfaffians are zero.) Therefore, by (5.1), $\operatorname{rad}(\mathfrak{C}) = \operatorname{rad}(\mathfrak{B}) = \operatorname{rad}(\mathfrak{A})$. The rest now follows from (2.1) and (5.4). $\square$

Remark 5.6. According to (4.4)(1), $\mathfrak{A}$ is the same as the ideal $\mathfrak{A}'$ generated by *all* sub-Pfaffians $\operatorname{Pf}(A[J_1])$ with $J_1 \supseteq \{1, \ldots, p\}$. The point of this statement is that, while we can only say that $\mathfrak{B} \subseteq \mathfrak{B}'$ have the same radicals in (5.1), (4.4)(1) actually yields the equality $\mathfrak{A} = \mathfrak{A}'$ — *without* taking the radicals.

Example 5.7. In the case $p = 1$, we have $\mathfrak{A} = \mathfrak{C}$, so the conclusion of (5.5) is clear. For $p = 2$, take $A \in \mathbb{A}_4(R)$ as in (7.3) below. Writing $\langle a_1, a_2, \ldots \rangle$ for the ideal in R generated by the elements a_i's, we have by definition $\mathfrak{A} = \langle x, \operatorname{Pf}(A) \rangle$, $\mathfrak{B} = \langle x^2, \operatorname{Pf}(A)^2 \rangle$ (with $\operatorname{Pf}(A) = xw - yv + zu$), while

$$\mathfrak{C} = \langle x^2, xy, xu, xz, xv, yv - zu \rangle .$$

Quick inspection shows that $\mathfrak{A}^3 \subseteq \mathfrak{B} \subseteq \mathfrak{A}^2 \subseteq \mathfrak{C} \subseteq \mathfrak{A}$. This clearly implies that the three ideals $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ have the same radical.

6. Some Determinantal Identities

This section is devoted to the proof of two determinantal identities involving matrices in $\mathbb{A}_n(R)$, which will be applied to the study of alternating-clean matrices. Throughout, R continues to denote an arbitrary commutative ring.

Lemma 6.1. *Let $M = r \cdot u^T u$, where $r \in R$, $u = (u_1, \ldots, u_n) \in R^n$, and n is even. For any $A \in \mathbb{A}_n(R)$, we have* $\det(M + A) = \det(A)$.

Proof. To prove this, it suffices to work with the case where r, $u_1, \ldots, u_n$ and the above-diagonal entries of A are commuting indeterminates. Let R' be the polynomial ring generated by these indeterminates over $\mathbb{Z}$, and let K be the quotient field of R'. Over K, we have $u \cdot G = (1, 0, \ldots, 0)$ for some $G \in \mathrm{GL}_n(K)$. We need only prove the determinantal identity with M replaced by $G^T MG$ and A replaced by $G^T AG$. But

$$G^T MG = r \cdot (uG)^T (uG) = r \cdot E_{11} \ \text{ (where } E_{ij} \text{ are the matrix units),}$$

and $G^T AG$ is again alternating (by [Al: Th. 1, p. 389]). Since n is even, the cofactor of $G^T AG$ at the (1,1)-position is zero. Computing determinants by expansion along the first row shows that $\det(r \cdot E_{11} + G^T AG) = \det(G^T AG)$, as desired. □

Remark 6.2. Clearly, the Lemma does not hold for odd n, as it fails badly enough for $n = 1$. Note that for $u = (1, \ldots, 1)$, $M = r \cdot u^T u$ is the matrix all of whose entries are r. In this case, the Lemma says that, if we add a constant r to all entries of an *even-sized* alternating matrix A (over any commutative ring), the determinant of A remains unchanged. This is a classical result, which can be found in [Mu: p. 397]. However, we have not been able to find the more general version (6.1) in [Mu] or in other books on determinant theory.

The next result extends (6.1) a little bit more, with an application to $\mathbb{A}$-cleanness. Recall that a matrix $M \in \mathbb{M}_n(R)$ is *alternating-clean* (or $\mathbb{A}$-clean for short) if $M = A + U$ where $A \in \mathbb{A}_n(R)$ and $U \in \mathrm{GL}_n(R)$; see [LS].

Theorem 6.3. *Let n be an even integer, and let $M \in \mathbb{M}_n(R)$ be a symmetric matrix with $\mathcal{D}_2(M) = 0$. Then, for any $A \in \mathbb{A}_n(R)$, we have* $\det(M+A) = \det(A)$, *and the matrix M is $\mathbb{A}$-clean.*

Proof. The second conclusion follows easily from the first as follows. Let A be the hyperbolic matrix in $\mathbb{A}_n(R)$ (see Remark 2.12). Then $\det(A) = 1$, and the first conclusion gives $\det(M + A) = 1$. Thus,

$$M = (-A) + (M + A) \in \mathbb{A}_n(R) + \mathrm{SL}_n(R)$$

shows that M is $\mathbb{A}$-clean.

To check the first conclusion, we apply an embedding technique similar to that used in the proof of (3.4)(3). Let S be the polynomial domain $\mathbb{Z}[y_{ij}]$ (where $1 \le i < j \le n$), and set $y_{ii} = 0$ and $y_{ji} = -y_{ij}$ for $j > i$. Further, let $R' := S[x_{ij}]$ with the relations

$$(6.4) \qquad x_{ij} = x_{ji}, \quad \text{and} \quad x_{ij}x_{k\ell} = x_{i\ell}x_{kj}$$

for all indices i, j, k, ℓ. It will suffice to handle the "generic" case where $M = (x_{ij})$ and $A = (y_{ij})$ over the ring R'. We make the following

Claim 6.5. *R' is a domain, and $x_{11} \neq 0$.*

Given this, we can work in the quotient field K of R'. Taking $r := x_{11}^{-1} \in K$ and letting $u := (x_{11}, ..., x_{1n})$, the symmetry of M gives $M = r \cdot u^T u$, so (6.1) gives the desired result. (Indeed, for this argument to work, all we need is that x_{11} is a non 0-divisor in the ring R'. This would give an *embedding* of R' into its localization $R'' = R'[x_{11}^{-1}]$, and M has the desired form $x_{11}^{-1} \cdot u^T u$ over R''.)

To see that R' is a domain, we use the commutative monoid X generated by the x_{ij} 's with the relations in (6.4), so that R' is just the monoid ring $S[X]$. (This shows, in particular, that $x_{11} \neq 0$ in R'.) For the monoid map $f : X \to \mathbb{Z}^n$ well-defined by $f(x_{ij}) = e_i + e_j$ (where $\{e_h\}$ are the standard unit vectors), we'll prove the following.

Lemma 6.6. *f is injective, and its image is the set of $a = (a_1, ..., a_n) \in \mathbb{Z}^n$ with all $a_h \geq 0$ and $\sum a_h$ even.*

Expressed in *multiplicative* notations, this lemma shows that the S-algebra homomorphism $R' \to S[z_1, ..., z_n]$ defined by $x_{ij} \mapsto z_i z_j$ is *injective.* From this, it follows that R' is a domain as claimed.

Proof of (6.6). Clearly, $\mathrm{im}(f)$ is in the given set. Conversely, if $a = (a_1, \ldots, a_n) \neq 0$ is in the given set, then either some $a_i \geq 2$ or for some distinct i, j, $a_i \geq 1$ and $a_j \geq 1$. By looking at $a - f(x_{ii})$ (resp. $a - f(x_{ij})$), it follows by induction on $\sum a_h$ that $a \in \mathrm{im}(f)$.

Suppose $f(x) = f(y) = a$, and say $a_i \geq 1$ and $a_j \geq 1$ where $i \neq j$. If x does not have a factor x_{ij}, it must have a factor $x_{i\ell} x_{kj} = x_{ij} x_{k\ell}$, so in any case $x = x_{ij}\, x'$ and similarly $y = x_{ij}\, y'$. Since $f(x') = f(y')$, it follows by induction on $\sum a_h$ that $x' = y'$, and therefore $x = y$. The same argument with $j = i$ applies if some $a_i \geq 2$. (Note that this argument can also be used to give a simpler proof of Prop. 13.2 in the third author's paper [Sw] on Gubeladze's theorem.) □

Remark 6.7. In (6.3), the assumption of *symmetry* on M (in addition to $\mathcal{D}_2(M) = 0$) is essential for its $\mathbb{A}$-cleanness. The following example illustrating this point is noteworthy. Consider the 16 matrices in $\mathbb{M}_2(\mathbb{F}_2)$. The six matrices in $\mathrm{GL}_2(\mathbb{F}_2)$ are, of course, $\mathbb{A}$-clean. The other ten matrices have rank ≤ 1, so the four *symmetric* ones (namely, E_{11}, E_{22}, $E_{11} + E_{12} + E_{21} + E_{22}$, and the zero matrix) are $\mathbb{A}$-clean, as is predicted by (6.3) (or already by (6.1)). The remaining six *nonsymmetric* matrices turned out to be *not* $\mathbb{A}$-clean. In fact, these are exactly the six exceptional matrices over $\mathbb{F}_2$ in Theorem B of [LS] classifying $\mathbb{A}$-clean matrices over all fields!

For more applications to $\mathbb{A}$-cleanness in the next two sections, we'll need another determinantal identity. This one is most probably known, but again, we have not been able to find it in the literature. Recall (from §4) that for a matrix $A = (a_{ij}) \in \mathbb{M}_n(R)$ and any subset $J \subseteq \{1, \ldots, n\}$, J' denotes $\{1, \ldots, n\} \setminus J$, and $A[J]$ denotes the submatrix of A consisting of the entries a_{ij} for which $i, j \in J$.

Theorem 6.8. *Let D, $A \in \mathbb{M}_n(R)$, where D is diagonal. Then*

$$\det(D + A) = \sum \det(D[J']) \cdot \det(A[J]), \tag{6.9}$$

where J ranges over all subsets of $\{1, \dots, n\}$. (Here, we use the convention that the determinant of an "empty matrix" is 1.)

Proof. If $D = \lambda I_n$, this boils down to the classical theorem that the coefficient of λ^{n-i} in the characteristic polynomial of A is $(-1)^i$ times the sum of the $i \times i$ principal minors of A; see, e.g. [MM: Ch. I, (2.13.2)]. In particular, (6.9) holds in the case $D = I_n$. To prove (6.9) in general, it is sufficient to work in the case where the diagonal elements of D and the entries of A are (commuting) indeterminates. Thus, we may replace R by the rational function field over $\mathbb{Q}$ generated by these indeterminates, over which D becomes *invertible.* In this case, (6.9) follows easily from the case we have covered, in view of the identity

$$\det(D + A) = \det(D) \cdot \det(I_n + D^{-1}A). \qquad \square$$

We shall apply (6.9) primarily in the case where A is alternating. In this case, the principal submatrices $A[J]$ are also alternating, so in (6.9) we may replace $\det(A[J])$ by $\mathrm{Pf}(A[J])^2$, and drop all terms with $|J|$ odd. With these steps, (6.9) "simplifies" to the following, which will have nice applications in the next two sections.

Corollary 6.10. *Let $A \in \mathbb{A}_n(R)$. Then for any diagonal matrix $D \in \mathbb{M}_n(R)$,*

$$\det(D + A) = \sum \det(D[J']) \cdot \mathrm{Pf}(A[J])^2, \tag{6.11}$$

where J ranges over all subsets of $\{1, \dots, n\}$ with even cardinality.

7. Examples of $\mathbb{A}$-Clean Diagonal Matrices

This section is devoted to a study of examples of $\mathbb{A}$-clean diagonal matrices over a commutative ring. We start with the relatively easy cases $n = 2, 3$ over the ring of integers $\mathbb{Z}$.

Proposition 7.1. *For $a, b, c \in \mathbb{Z}$, the following holds.*

(1) *The matrix $D = \mathrm{diag}(a, b)$ is $\mathbb{A}$-clean over $\mathbb{Z}$ iff one of $-ab \pm 1$ is a perfect square.*

(2) *The matrix $D = \mathrm{diag}(a, b, c)$ is $\mathbb{A}$-clean over $\mathbb{Z}$ iff $az^2 + by^2 + cx^2$ represents one of $-abc \pm 1$.*

Proof. Deciding the $\mathbb{A}$-cleanness of D amounts to checking the existence of an alternating matrix $A = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$ such that $D + A$ has a unit determinant. In the present case, (6.11) boils down to $\det(D + A) = ab + x^2$. This gives (1). Similarly, (2) follows by noting that, for $A = \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix}$, (6.11) amounts to

$$\det(D + A) = abc + az^2 + by^2 + cx^2. \qquad \square$$

Examples 7.2. Over $R = \mathbb{Z}$, we offer the following examples where the diagonal matrix D has entries of mixed signs.

(1) $\mathrm{diag}(-1, n^2 \pm 1)$ (for any n) and $\mathrm{diag}(2, -4)$ are $\mathbb{A}$-clean, but $\mathrm{diag}(2, -3)$ and $\mathrm{diag}(a, -a)$ (for any $a \geq 2$) are not.

(2) $\operatorname{diag}(0,2,-3)$ and $\operatorname{diag}(1,2,-3)$ are $\mathbb{A}$-clean. (This follows from (7.1)(2), by taking (z,y,x) to be $(0,1,1)$ and $(0,4,3)$ respectively.)

(3) If $\gcd(a,b,c) \neq 1$, (7.1)(2) shows that $\operatorname{diag}(a,b,c)$ is *not* $\mathbb{A}$-clean. A less trivial non $\mathbb{A}$-clean 3×3 example of mixed signs (and with $\gcd(a,b,c) = 1$) is $\operatorname{diag}(0,5,-3)$. (Working modulo 5 shows that $5y^2 - 3x^2 \neq \pm 1$.) This example serves to show, for instance, that the notion of $\mathbb{A}$-cleanness of matrices does not satisfy a "local-global principle". Indeed, consider any localization of $\mathbb{Z}$ at a prime ideal $\mathfrak{p}$. If $\mathfrak{p} = (0)$, A is certainly $\mathbb{A}$-clean over $\mathbb{Z}_{\mathfrak{p}} = \mathbb{Q}$, by [LS: Th. B]. If $\mathfrak{p} = (p)$ for a prime number p, the reduction of A to the residue class field of $\mathbb{Z}_{(p)}$ is not alternating (one of the diagonal entries doesn't reduce to zero), so A is again $\mathbb{A}$-clean over the local ring $\mathbb{Z}_{(p)}$, according to [LS: Cor. 6.2]. And yet, A itself is not $\mathbb{A}$-clean over $\mathbb{Z}$. (Counterexamples like this also exist for 2×2 matrices.)

Next, we go to the case $n = 4$, which, incidentally, provides the first truly nontrivial illustration of the determinantal formula (6.11) over a commutative ring. Let us first write out (6.11) more explicitly, using the notations

$$D = \operatorname{diag}(a,b,c,d), \quad \text{and} \quad A = \begin{pmatrix} 0 & x & y & u \\ -x & 0 & z & v \\ -y & -z & 0 & w \\ -u & -v & -w & 0 \end{pmatrix} \in \mathbb{A}_4(R). \tag{7.3}$$

For $|J| = 0$ and $|J| = 4$ respectively, we get the terms $\det(D) = abcd$ and $\det(A) = \big(\operatorname{Pf}(A)\big)^2 = (xw - yv + zu)^2$ on the RHS of (6.11). On the other hand, if $|J| = 2$, we have the possibilities

$$J = \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,4\},\ \{2,4\},\ \text{and } \{3,4\},$$

which, together, contribute the sum $cdx^2 + bdy^2 + adz^2 + bcu^2 + acv^2 + abw^2$ to the RHS of (6.11). Therefore, (6.11) yields

$$\begin{aligned} \det(D + A) = abcd + cdx^2 + bdy^2 + adz^2 + bcu^2 + acv^2 + abw^2 \\ + (xw - yv + zu)^2. \end{aligned} \tag{7.4}$$

This determinantal formula in the 4×4 case can be found, for instance, on p. 410 of Muir's book [Mu]. In view of this, the diagonal matrix $D = \operatorname{diag}(a,b,c,d)$ is $\mathbb{A}$-clean iff the polynomial on the RHS of (7.4) represents a unit of R. This fact leads to the following interesting concrete examples of $\mathbb{A}$-clean matrices, which contrast sharply with the examples (over $\mathbb{Z}$) in (7.2)(3).

Corollary 7.5. *Let $a, b \in R$, where R is a commutative ring. Then the following diagonal matrices are $\mathbb{A}$-clean:*

$$\operatorname{diag}(a,b,0,-a), \quad \operatorname{diag}(a,a,a,-a), \quad \textit{and} \quad \operatorname{diag}(a,a,-a,-a).$$

Proof. (1) Here, $c = 0$ and $d = -a$. Taking $(x,y,z) = (1,1,0)$ and $(u,v,w) = (0,0,1)$ makes the RHS of (7.4) equal to 1.

(2) Here, $b = c = a$ and $d = -a$. Taking $(x,y,z) = (1,0,0)$ and $(u,v,w) = (a,0,1)$ makes the RHS of (7.4) equal to 1.

(3) Here, $b = a$ and $c = d = -a$. Taking $(x,y,z) = (a+1,a,a)$ and $(u,v,w) = (a,a,a-1)$ makes again the RHS of (7.4) equal to 1. $\square$

We note in passing that, although the above proof for (7.5) made use of determinants, the commutativity of R is not really essential. For the alternating matrices A constructed in this proof, it is a simple matter to check that $D + A$ is in fact invertible without any commutativity assumptions. This is trivial in the cases (2) and (3), and even in the case (1), elementary row and column operations can be used to show directly that $D + A \in \mathrm{GL}_4(R)$ without assuming $ab = ba$.

8. Non-Negative Diagonal Matrices Over $\mathbb{Z}$

In this section, we focus on the ring $\mathbb{Z}$. Our study in §7 showed that there are many examples of $\mathbb{A}$-clean diagonal matrices over $\mathbb{Z}$ with entries of mixed signs. However, it also hinted at the possible scarcity of such matrices with *non-negative* entries. For instance, $A = \mathrm{diag}\,(0, 1, 2, 2) \in \mathbb{M}_2(\mathbb{Z})$ is *not* $\mathbb{A}$-clean since for these choices of a, b, c, d, the RHS of (7.4) becomes

$$4x^2 + 2y^2 + 2u^2 + (xw - yv + zu)^2,$$

which clearly does not represent ± 1 over $\mathbb{Z}$. The matrix A here provides another example for the failure of the local-global principle for $\mathbb{A}$-cleanness, as it is $\mathbb{A}$-clean over all local rings, according to [LS: Cor. 6.2].

The example above suggests that it might be possible to apply (6.11) to get a *complete* classification of $\mathbb{A}$-clean diagonal matrices over $\mathbb{Z}$ with nonnegative entries. This work is successfully carried out in Theorem 8.1 below. However, the proof of this classification theorem depends also in a crucial way on the Pfaffian Vanishing Theorem 4.4. Indeed, it was the need to consummate the proof of (8.1) that had led us first to suspect, and then to prove, the general truth of (4.4). This in turn inspired the Pfaffian Expansion Theorem 4.2. Thus, (4.2) was very much a pleasant case of a mathematical result that was discovered through the working with a specific class of examples.

Theorem 8.1. *Let* $D = \mathrm{diag}\,(d_1, \ldots, d_n) \in \mathbb{M}_n(\mathbb{Z})$*, where all* $d_i \geq 0$*. Then* D *is* $\mathbb{A}$*-clean iff, up to a permutation,* $(d_1, \ldots, d_n)$ *has the form*

$$(0, *\,; \ldots ; 0, *\,; 1, \ldots, 1). \tag{8.2}$$

Here, the $*$*'s denote (possibly different) non-negative integers. (In particular, if all* $d_i > 0$*,* D *is* $\mathbb{A}$*-clean iff* $D = I_n$*.)*

Proof. The "if" part above is true over any ring R, without any assumptions on the diagonal entries d_i. (In this case, the $*$'s denote arbitrary elements of R.) To see this, we simply note that a matrix of the form $\mathrm{diag}\,(0, *)$ is always $\mathbb{A}$-clean, via the decomposition

$$\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & * \end{pmatrix}.$$

Since $\mathrm{diag}\,(1, \ldots, 1)$ is invertible (and hence also $\mathbb{A}$-clean), the "if" part of the theorem follows by forming orthogonal sums.

For the "only if" part, assume D is $\mathbb{A}$-clean (over $\mathbb{Z}$), and fix a matrix $A \in \mathbb{A}_n(\mathbb{Z})$ such that $D + A \in \mathrm{GL}_n(\mathbb{Z})$. Since all $d_i \geq 0$, the determinantal formula (6.11) implies that $\det(D+A)=1$, and that there exists $J_1 \subseteq \{1,\dots,n\}$ (necessarily with $|J_1|$ even) such that

$$\det(D\,[J_1'\,]\,)=1 \quad \text{and} \quad \operatorname{Pf}(A\,[J_1]\,)^2=1. \tag{8.3}$$

The first equation implies that J_1 must contain $J_0 := \{\, i \in [1,n] :\ d_i = 0\}$ (since otherwise J_1' would contain some $i \in J_0$, which would have made $\det(D[J_1'\,])=0$). Letting $p=|J_0|$, we may assume, for convenience, that $J_0=\{1,\dots,p\}$. We go into the following two cases.

Case 1. $n \le 2p$. If n is even, the desired conclusion is trivial. If n is odd, we need to show that at least one d_i is 1. This is clear since $|J_1'|=n-|J_1|>0$, and $i \in J_1'$ implies $d_i=1$.

Case 2. $n > 2p$. Here, we finish by proving that:

(8.4) *there are at least* $n-2p$ *ones among the* d_i*'s.*

If this is the case, we can clearly permute $(d_1,\dots,d_n)$ into the form (8.2). To prove (8.4), we apply (4.4)(2). Since $\operatorname{Pf}(A\,[J_1]\,)\neq 0$, (4.4)(2) implies that $\operatorname{Pf}(A\,[J\,]\,)\neq 0$ for some $J \supseteq J_0$ with $|J\,| \le 2p$. Then $\det(D\,[J\,']\,)\neq 0$ also (as $J\,' \cap J_0=\emptyset$), and we have necessarily $i \in J\,' \Rightarrow d_i=1$. (In fact, we must have $J=J_1$, but this is not needed.) Since $|J\,'\,|=n-|J\,| \ge n-2p$, this proves (8.4). $\square$

References

[Al] A. A. Albert: *Symmetric and alternate matrices in an arbitrary field.* Trans. Amer. Math. Soc. **43** (1938), 386–436.

[Ar] E. Artin: *Geometric Algebra.* Interscience Publishers, New York, 1957.

[BS] G. Boffi and R. Sánchez: *Some classical formulas and a determinantal ideal.* Seminari di Geometria, Bologna 1988–91, pp. 27–35, Univ. Stud. Bologna, 1991.

[BC] W. Bruns and A. Conca: *Gröbner bases and determinantal ideals.* Commutative Algebra, Singularities and Computer Algebra (Sinaia, 2002), 9–66, NATO Sci. Ser. II Math. Phys. Chem., **115**, Kluwer Acad. Publ., Dordrecht, 2003.

[BV] W. Bruns and U. Vetter: *Determinantal Rings.* Lecture Notes in Mathematics, Vol. **1327**, Springer-Verlag, Berlin-Heidelberg-New York, 1988.

[Ca] A. Cayley: *Sur les déterminants gauches.* Crelle J. **38** (1854), 93–96.

[Co] A. Conca: *Gröbner bases of ideals of minors of a symmetric matrix.* J. Algebra **106** (1994), 406–421.

[Fr] G. Frobenius: *Über das Pfaff'sche Problem.* Crelle J. **82** (1876), 230–315.

[He] P. Heymans: *Pfaffians and skew-symmetric matrices.* Proc. London Math. Soc. **19** (1969), 730–768.

[HoS] O. Holtz and B. Sturmfels: *Hyperdeterminantal relations among symmteric principal minors.* `http://arxiv.org/abs/math.RA/0604374`, January, 2007.

[HS] R. A. Horn and V. V. Sergeichuk: *Regularization algorithm for matrices of bilinear and sesquilinear forms.* Linear Alg. Appl. **412** (2006), 380–395.

[Jo] T. Jósefiak: *Ideals generated by minors of a symmetric matrix.* Comment. Math. Helv. **53** (1978), 595–607.

[JP] T. Jósefiak and P. Pragacz: *Ideals generated by Pfaffians.* J. Algebra **61** (1979), 189–198.

[JPW] T. Jósefiak, P. Pragacz, and J. Weyman: *Resolutions of determinantal varieties and tensor complexes associated with symmetric and antisymmetric matrices.* Young Tableaux and Schur Functors in Algebra and Geometry (Toruń, 1980), pp. 109–189, Astérisque, 87–88, Soc. Math. France, Paris, 1981.

[Ka] I. Kaplansky: *Commutative Rings.* University of Chicago Press, 1974.

[Ku] K. Kurano: *Relations on Pfaffians. I. Plethysm formulas.* J. Math. Kyoto Univ. **35** (1995), 495–533.

[Kut] R. E. Kutz: *Cohen-Macaulay rings and ideal theory in rings of invariants of algebraic groups.* Trans. Amer. Math. Soc. **194** (1974), 115–129

[La_1] T. Y. Lam: *A general theory of Vandermonde matrices.* Expos. Math. **4** (1986), 193–215.

[La_2] T. Y. Lam: *On the equality of row rank and column rank.* Expos. Math. **18** (2000), 161–164.

[La_3] T. Y. Lam: *Exercises in Classical Ring Theory.* Second Ed., Problem Books in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 2003.

[La_4] T. Y. Lam: *Serre's Problem on Projective Modules.* Monographs in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 2006.

[LS] T. Y. Lam and R. G. Swan: *Sums of alternating matrices and invertible matrices.* Algebra and Its Applications, Proc. International Conference, Athens, Ohio, March, 2005, D.V. Huynh, S.K. Jain, S. López-Permouth, eds, Contemp. Math. **419** (2006), 201–211, Amer. Math. Soc., Providence, R.I.

[MM] M. Marcus and H. Minc: *A Survey of Matrix Theory and Matrix Inequalities.* Allyn and Bacon, Inc., Boston, Mass., 1964. (Reprinted by Dover Publications, Inc., New York, 1992.)

[Mc] N. McCoy: *Rings and Ideals.* Carus Math. Monographs, Vol. 8, Math. Assoc. of America, 1948.

[Mu] T. Muir: *A Treatise on the Theory of Determinants.* Longmans, Green and Co., 1933. (Reprinted by Dover Publications, Inc., New York, 1960.)

[S] R. Stoll: *Linear Algebra and Matrix Theory.* McGraw Hill Co. Inc., 1952. (Reprinted by Dover Publications, Inc., New York, 1969.)

[St] B. Sturmfels: *Gröbner bases and Stanley decompositions of determinantal rings.* Math. Zeit. **205** (1990), 137–144.

[Sw] R. G. Swan: *Gubeladze's proof of Anderson's conjecture.* Azumaya Algebras, Actions, and Modules (Bloomington, Indiana, 1990), Contemp. Math. **124** (1992), 215–250, Amer. Math. Soc., Providence, R.I.

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