Math 172 - Problem Set 5 - due Wednesday, November 23

Same rules as last time.

1. (a) Given a set $S \subset \mathbb{R}^d$, prove that $\mathrm{conv}(S)$ is convex.
   (b) Let $C \subset \mathbb{R}^d$ be a convex set, and let S be a subset of C. Prove that $\mathrm{conv}(S) \subset C$.

2. Let $P_1 \subset \mathbb{R}^{d_1}$ and $P_2 \subset \mathbb{R}^{d_2}$ be polytopes, and define $P_1 \times P_2$ to be the Cartesian product $\{(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} : \ x \in P_1, y \in P_2\}$.
   (a) If F_1 is a face of P_1 and F_2 is a face of P_2, prove that $F_1 \times F_2$ is a face of $P_1 \times P_2$. In fact, one can show (you don't have to) that these are the only faces of $P_1 \times P_2$.
   (b) Let P_1 be a triangle and P_2 be a pentagon.
      i. How many facets does $P_1 \times P_2$ have and what type are they (tetrahedra, cubes, prisms, pyramids, etc.)?
      ii. How many 2-faces does $P_1 \times P_2$ have and what type are they (triangles, quadrilaterals, pentagons, etc.)?
      iii. How many edges and how many vertices does $P_1 \times P_2$ have?

3. Recall that we can write the d-cube as $[-1, 1]^d$, the Cartesian product of $[-1, 1]$ with itself d times. We also know that $[-1, 1]$ is a polytope with two 0-dimensional faces (the vertices $+1$ and -1) and one 1-dimensional face (the whole interval $[-1, 1]$). Using these facts, and using the results from Question 2, count how many i-dimensional faces a d-cube has, for $0 \leq i \leq d$.

4. Show that two 3-dimensional polytopes with the same f-vector can be "different." In particular, find two 3-dimensional polytopes, draw their Schlegel diagrams, and show that these diagrams (thought of as graphs) are not isomorphic. Hint: look at the constructions we used to construct polytopes with a given f-vector.

5. Let P be the cyclic polytope $C(d, n)$ in d dimensions with n vertices.

(a) Let $I = \{i_1, i_2, \ldots, i_d\} \subset \{1, 2, \ldots, n\}$ be such that, for all j_1 and j_2 in $\{1, 2, \ldots, n\} \setminus I$, the number of integers

$$i \text{ such that } i \in I \text{ and } j_1 < i < j_2$$

is even. Prove that $\mathrm{conv}(v_{i_1}, v_{i_2}, \ldots, v_{i_d})$ is a facet of P. Hint: consider the polynomial $p(x) = (x - i_1)(x - i_2) \cdots (x - i_d)$; prove that either $p(j) < 0$ for all $j \notin I$ or $p(j) > 0$ for all $j \notin I \ldots$

Note: one can prove (you don't have to) that these are the only facets of P.

(b) How many facets does $C(4, 7)$ have?