Math 172 - Problem Set 3 - due Monday, October 17

Same rules as last time.

1. Let $\chi_G(x)$ be the chromatic polynomial for the graph G.

   (a) Prove that, if $G = C_n$, the cycle on n vertices, then

   $$\chi_G(x) = (x-1)\big((x-1)^{n-1} + (-1)^n\big).$$

   (b) Compute $\chi_G(k)$, where $G = W_n$, the "wheel graph" which has $n+1$ vertices and consists of an n vertex cycle whose vertices are each connected to a "center" vertex (to be exact, W_n is the graph with vertices $\{v_1, v_2, \ldots, v_n, w\}$ and with edges $\{v_i, v_{i+1}\}$ for $1 \le i \le n-1$, $\{v_n, v_1\}$, and $\{w, v_i\}$ for $1 \le i \le n$).

2. Prove that a graph is bipartite if and only if it contains no cycles of odd length.

3. If we draw $K_{4,4}$ in the plane (so that no edge passes through a vertex, but edges may cross eachother) what is the minimum number of edge crossings we can have?

4. Given a graph G, let

   $$\chi_G(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_{s+1} x^{s+1} + a_s x^s,$$

   where $a_d \neq 0$ and $a_s \neq 0$. Prove that:

   (a) $d = |V(G)|$,

   (b) $a_d = 1$,

   (c) $a_{d-1} = -|E(G)|$,

   (d) s is the number of connected components of G, and

   (e) if G contains any edges, then $\sum_{i=s}^{d} a_i = 0$.

5. The *complement* of a graph G is the graph with vertices $V(G)$ and edges

   $$\big\{\{u,v\} : \ \{u,v\} \notin E(G)\big\}.$$

   A graph which is isomorphic to its complement is called *self-complementary.*

(a) Prove that, if G is self-complementary, then the number of vertices of G is congruent to 0 or 1 modulo 4.

(b) For each n congruent to 0 or 1 modulo 4, construct an n-vertex graph which is self-complementary. (Hint: Do this for $n = 4$. Then, for larger n congruent to 0 modulo 4, place the vertices into four groups and somehow use your solution for $n = 4$.)