Math 172 - Problem Set 2 - due Monday, October 3

Same rules as last time, though please feel free to look back at your Calculus / Differential Equations textbooks.

1. Let a_n be defined by the recurrence $a_n = 3a_{n-1} + 2^n$, for $n \geq 1$, with a_0=1. Compute a_n as a function of n.

2. (a) Let a_n be defined by the recurrence $a_n = 6a_{n-1} - 8a_{n-2}$, for $n \geq 2$, with $a_0 = 0$ and $a_1 = 1$. Find the generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$, and use it to compute a_n as a function of n.

   (b) Find the exponential generating function $g(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ by solving a differential equation, and use it to compute a_n as a function of n.

   (c) (ungraded) Muse about how the *characteristic polynomial* $1-6x+8x^2$ shows up in parts a and b.

   (d) Let b_n be defined by the recurrence $b_n = 6b_{n-1} - 9b_{n-2}$, for $n \geq 2$, with $b_0 = 0$ and $b_1 = 1$. Repeat parts a and b for b_n.

   (e) (ungraded) Ruminate about how the fact that the characteristic polynomial $1 - 6x + 9x^2$ has a double root affects things.

3. Let $g(x)$ and $h(x)$ be two formal power series. Characterize when $\frac{g(x)}{h(x)}$ "makes sense," that is, when there exists a formal power series $a(x)$ such that $a(x)h(x) = g(x)$. (Your answer should be something like "$\frac{g(x)}{h(x)}$ makes sense if and only if ... ").

4. Given integers n, r, and s, define $p_{r,s}(n)$ to be the number of partitions of the integer n into parts that are at most r and into at most s parts (for example, $10 = 5+3+2$ is a partition of $n = 10$ into parts that are at most 5 and into at most 3 parts).

   (a) Given r and s, define the generating function

   $$F_{r,s}(x) = \sum_{n=0}^{\infty} p_{r,s}(n) x^n.$$

   Show that $F_{r,s}$ satisfies the recurrence

   $$F_{r,s}(x) = x^r F_{r,s-1}(x) + F_{r-1,s}(x),$$

   for $r, s \geq 1$.

(b) Given integers n and k, define the function

$$\begin{bmatrix} n \\ k \end{bmatrix}_x = \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k+1})}{(1-x^k)(1-x^{k-1})\cdots(1-x)}.$$

Prove that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_x$ satisfies the recurrence

$$\begin{bmatrix} n \\ k \end{bmatrix}_x = x^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_x + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_x,$$

for $n, k \geq 1$.

(c) Prove that $F_{r,s}(x) = \left[\begin{smallmatrix} r+s \\ r \end{smallmatrix}\right]_x$.

(d) Use the fact that

$$\lim_{x \to 1} \frac{1-x^a}{1-x^b} = \frac{a}{b}$$

to prove that

$$\lim_{x \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_x = \binom{n}{k}.$$

(Aside: this fact, plus the fact that people usually use the variable q instead of x in this context, means that these are often called the "q-binomial coefficients.")

(e) Compute and interpret $F_{r,s}(1)$, in light of parts c and d.

5. We saw in class, using generating functions, that the number of partitions of n into distinct parts equals the number of partitions of n into odd parts. Find a bijection between partitions of n into distinct parts and partitions of n into odd parts.