Math 172 - Problem Set 4 Solutions

Problem 1:

By Theorem 2.8 from Bogart, the number of such trees is the coefficient of $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ in

$$x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}.$$

This is the same as the coefficient of $x_1^{d_1-1} x_2^{d_2-1} \cdots x_n^{d_n-1}$ in

$$(x_1 + x_2 + \cdots + x_n)^{n-2}.$$

This is the multinomial coefficient

$$\frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}.$$

Problem 2:

As several people pointed out, we need to assume that $k > 0$. If $k = 0$, then the graph has no edges and so doesn't have a complete matching (unless $V_1 = \emptyset$).

It suffices to show that, for any $X \subset V_1$, $|X| \leq |R(X)|$ (where $R(X)$ is the set of neighbors of X). Then Hall's Marriage Theorem will imply that there is a complete matching of V_1 into V_2.

Indeed, let X be a subset of V_1. Let $E(X)$ be the set of edges with an endpoint in X, and let $E\big(R(X)\big)$ be the set of edges with an endpoint in $R(X)$. By the definition of $R(X)$, we have that $E(X) \subseteq E\big(R(X)\big)$ (since every edge in X goes to something in $R(X)$, and so

$$|E(X)| \leq \left|E\big(R(X)\big)\right|.$$

Since each vertex in X is contained in at least k edges by assumption, and since none of these edges go between two vertices in X (all of the edges go from X to something in V_2), we have that

$$|E(X)| \geq k \cdot |X|.$$

Similarly, since every vertex in $R(X)$ is contained in at most k edges, we have that

$$\left|E\big(R(X)\big)\right| \leq k \cdot |R(X)|.$$

Putting all of this together,

$$k \cdot |X| \leq |E(X)| \leq \left|E\big(R(X)\big)\right| \leq k \cdot |R(X)|\,.$$

Dividing by k, we get

$$|X| \leq |R(X)|\,,$$

as desired, and the problem follows by Hall's Marriage Theorem.

Problem 3:

Part a: We must show two things:

1. For all edges $u \to v$, $0 \leq f_M(u,v) \leq c(u,v)$ and
2. For all vertices v except the sink and source,

$$\sum_{u:\ u\to v \text{ an edge}} f(u,v) = \sum_{u:\ v\to u \text{ an edge}} f(v,u).$$

The first item is true, as $c(u,v) = 1$ for all of the edges, and the flow on each edge is always 0 or 1. For the second item, there are 4 cases:

1. $v \in V_1$ and v is in the matching. Then $f_M(s,v) = 1$ by the definition of f_M, and, since M is a matching, there is exactly 1 vertex u such that (v,u) is in the matching. Then $f_M(v,u) = 1$, and all of the other edges containing v (besides $s \to v$ and $v \to u$)) have flow 0. Then

$$\sum_{u:\ u\to v \text{ an edge}} f(u,v) = 1 = \sum_{u:\ v\to u \text{ an edge}} f(v,u).$$

2. $v \in V_1$ and v is not in the matching. Then $f_M(s,v) = 0$, and, in fact, all edges containing v have flow 0. Then

$$\sum_{u:\ u\to v \text{ an edge}} f(u,v) = 0 = \sum_{u:\ v\to u \text{ and edge}} f(v,u).$$

3. $v \in V_2$ and v is in the matching.Then $f_M(v,t) = 1$ by the definition of f_M, and, since M is a matching, there is exactly 1 vertex u such that (u,v) is in the matching. Then $f_M(u,v) = 1$, and all of the other edges containing v (besides $v \to t$ and $u \to v$) have flow 0. Then

$$\sum_{u:\ u\to v \text{ an edge}} f(u,v) = 1 = \sum_{u:\ v\to u \text{ an edge}} f(v,u).$$

4. $v \in V_2$ and v is not in the matching. Then $f_M(v,t) = 0$, and, in fact, all edges containing v have flow 0. Then

$$\sum_{u:\ u \to v \text{ an edge}} f(u,v) = 0 = \sum_{u:\ v \to u \text{ an edge}} f(v,u).$$

Part b: First we prove the forward direction. Suppose f_M is a maximal flow. We must show that M is a maximal matching. Note that the value of the flow f_M is the sum if the flows on the edges leading out of the source s, which is the sum

$$\sum_{v \in V_1 \cap M} 1 = |M|\,,$$

so the value of the flow f_M is the same as the value of the matching M. Suppose there were a greater matching M'. Then the flow $f_{M'}$ would have value $|M'|$, which is greater than $|M|$, the value of the flow f_M. This contradicts that f_M is a maximum flow. Therefore M is a maximum matching.

Now we prove the reverse direction. Suppose M is a maximum matching. We must show that f_M is a maximum flow. Let f be a maximum flow. Since the capacities of the flow graph are all integers, we may assume that the maximum flow f has all integer values. In particular, since each edge has capacity 1, $f(u,v)$ must be 0 or 1 for all edges of the graph. Let M' be the set of edges (v_1, v_2) such that $v_1 \in V_1$ and $v_2 \in V_2$ and $f(v_1, v_2) = 1$. We will show that M' is a matching. Indeed, let v_1 be a vertex in V_1. Since the flow into v_1 is at most 1 (there is only one edge $s \to v_1$ into v_1), there can only be one v_2 such that $f(v_1, v_2) = 1$; for all of the other v_2, we must have $f(v_1, v_2) = 0$. Therefore v_1 is in at most one edge of M'. Similarly, each $v_2 \in V_2$ is in at most one edge of M', and so M' is a matching. In addition, for each $v_1 \in V_1$ that is contained in an edge of M', we must have that $f(s, v_1) = 1$ so that the flow into v_1 is the same as the flow out of it. Similarly

- if $v_1 \in V_1$ is not contained in an edge of M' then $f(s, v_1) = 0$
- if $v_2 \in V_2$ is contained in an edge of M' then $f(v_2, t) = 1$, and
- if $v_2 \in V_2$ is not contained in an edge of M' then $f(v_2, t) = 0$.

This means that f is exactly the flow $f_{M'}$. Since M is a maximal matching, $|M| \geq |M'|$, which means that f_M has as great a flow as $f = f_{M'}$. But since f is a maximal flow, this must mean that f_M is also a maximal flow. We're done.

Part c: Let P be an M-augmenting path. Then P starts and ends at unmatched vertices and alternates between edges not in the matching and edges in the matching. In particular, this means that the length of P is odd, so one of its endpoints is in V_1 and the other is in V_2. Without loss of generality, assume that P starts at some $v_1 \in V_1$ and ends at some $v_2 \in V_2$ (if it is the other way around, reverse the path P). We claim that the path

$$P' = (s \to v_1)P(v_2 \to t)$$

is a flow augmenting path for f_M. To show this, we must show that the flow on each forward edge in P' is under capacity and the flow on each backward edge in P' is non-zero. The first edge (s, v_1) is a forward edge, and since v_1 is not matched by M,

$$f_M(s, v_1) = 0 < 1 = c(s, v_1).$$

Now we traverse P. The ith edge of P has two possibilities. If i is odd, then the edge is a forward edge; also, the edge is not in the matching, so its flow is zero, which is under capacity. If i is even, then the edge is a backward edge; also, the edge is in the matching, so its flow is one, which is non-zero. The final edge in P', the edge (v_2, t) is a forward edge, and since v_2 is not in the matching, $f_M(v_2, t) = 0$, and the flow on that edge is under capacity. Therefore f_M is a flow augmenting path.

Problem 4:

At a given point in the hiring process, let R be the set of all (i, j) such that Employer i has been rejected by Applicant j. Since at each turn at least one employer is rejected by at least one applicant (if this doesn't happen, the algorithm terminates), the size of R must increase by at least 1 in every turn. But $|R|$ can be at most n^2, so the algorithm must terminate in at most n^2 turns.

Suppose, seeking a contradiction, that the matching resulting from this algorithm is not stable. Then there is an Employer i and Applicant j such that Employer i prefers Applicant j to the applicant she hired, and Applicant j prefers Employer i to the employer he went to work for. At each step, an Employer offers the job to the person she most prefers among the people who haven't yet rejected her. In particular, this means that if an employer offers a job to someone, she has already been rejected by every applicant she likes better. Therefore Employer i must have offered a job to Applicant j and been rejected. But we see, inductively, that Applicant j eventually accepts the offer that is his favorite, among all of the job offers he gets: if he gets a new offer at a stage in the process, then he either rejects the new offer if his previous favorite is more attractive, or he rejects his previous favorite if the new offer is more attractive. Therefore Applicant j accepted his favorite job, among all that he was offered. This contradicts the fact that he was offered a job by Employer i that he liked better than what he accepted. Therefore the resulting matching must be stable.

Problem 5:

Order the vertices of $K_{r,s}$ $v_i, v_2, \ldots, v_r, v_1', v_2', \ldots, v_s'$, where $v_i \in V_1$ and $v_i' \in V_2$. As defined in the Matrix-Tree Theorem, let A be the adjacency matrix for the graph, which is the block matrix

$$\begin{pmatrix} 0_{r,r} & 1_{r,s} \\ 1_{s,r} & 0_{s,s} \end{pmatrix},$$

where $a_{n,m}$ denotes the $n \times m$ matrix whose entries are all a (since each v_i is adjacent to each v_j'). Let B be the degree matrix, which is

$$\begin{pmatrix} s \cdot I_r & 0_{r,s} \\ 0_{s,r} & r \cdot I_s \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. Then let M be the matrix $B - A$, and let M_1 be the matrix formed by removing the first row and column of M, so

$$M_1 = \begin{pmatrix} s \cdot I_{r-1} & (-1)_{r-1,s} \\ (-1)_{s,r-1} & r \cdot I_s \end{pmatrix}.$$

Then by the Matrix-Tree Theorem, the number of spanning trees of $K_{r,s}$ is the determinant of M_1. To calculate the determinant of M_1 we may perform elementary row and columns operations, neither of which change the determinant. For $1 \leq i \leq r-1$, the ith row has one "s" and s "-1"s, so the sum of all of the elements in the ith row is 0. For $r \leq i \leq r+s-1$, the ith row has one "r" and $r-1$ "-1"s, so the sum of all of the elements in the ith row is 1. Therefore, if we replace the last column of M_1 with the sum of all of the columns, we get the new matrix

$$M' = \begin{pmatrix} s \cdot I_{r-1} & (-1)_{r-1,s-1} & 0_{r-1,1} \\ (-1)_{s-1,r-1} & r \cdot I_{s-1} & 1_{s-1,1} \\ (-1)_{1,r-1} & 0_{1,s-1} & 1 \end{pmatrix}.$$

Now if, for $1 \leq i \leq r-1$, we replace the ith column with the sum of the ith column and the last column, we get the matrix

$$M'' = \begin{pmatrix} s \cdot I_{r-1} & (-1)_{r-1,s-1} & 0_{r-1,1} \\ 0_{s-1,r-1} & r \cdot I_{s-1} & 1_{s-1,1} \\ 0_{1,r-1} & 0_{1,s-1} & 1 \end{pmatrix}.$$

This matrix M'' is upper diagonal, so its determinant is the product of its diagonal entries. Therefore

$$\det(M_1) = \det(M'') = s^{r-1} r^{s-1} \cdot 1 = s^{r-1} r^{s-1}$$

is the number of spanning trees of $K_{r,s}$.