Math 172 - Problem Set 1 Solutions

Problem 1:

Let A be the set of partitions of the integer m into n distinct parts. Let B be the set of partitions of m into n distinct parts, none of which are 1. Let C be the set of partitions of m into n distinct parts, exactly one of which is 1. A is the disjoint union of B and C (since the parts must be distinct, there is no partition with more than one 1).

We will show that there is a bijection from the set B to the set of partitions of $m-n$ into n distinct parts, and a bijection from the set C to the set of partitions of $m-n$ into $n-1$ distinct parts. This will imply that

$$Q(m,n) = |A| = |B| + |C| = Q(m-n,n) + Q(m-n,n-1),$$

as desired.

Indeed, given a partition $(\lambda_1, \lambda_2, \dots, \lambda_n)$ from B (with $\lambda_1 + \cdots + \lambda_n = m$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 1$, since the parts are distinct and none of them are 1), map it to the partition $(\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_n - 1)$. This is a partition of $m-n$ into n parts (none of the $\lambda_i - 1$ are zero), and the parts are still distinct. There is an inverse map, adding 1 to each part of the partition, so this is a bijection from B to the set of partitions of $m-n$ into n distinct parts.

Similarly, given a partition $(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 1)$ from C (with $\lambda_1 + \cdots + \lambda_{n-1} + 1 = m$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} > 1$), map it to the partition $(\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_{n-1} - 1)$. This is a partition of $m-n$ into $n-1$ parts, and the parts are distinct. There is an inverse map: if $(\mu_1, \dots, \mu_{n-1})$ is a partition of $m-n$ into $n-1$ parts, then the inverse map takes it to $(\mu_1 + 1, \dots, \mu_{n-1} + 1, 1)$, which is in C. Therefore this is a bijection from C to the set of partitions of $m-n$ into $n-1$ distinct parts, and we are done!

Problem 2

a. First we need to resolve something that's ambiguous in the statement of the problem: if a child puts shoe A on his left foot and shoe B on his right foot, is that the same as putting shoe B on his left foot and shoe A on his right? Let's say they are different situations.

We will use Inclusion-Exclusion. Let's number the children Child 1 through Child m. For $1 \leq j \leq m$ let p_j be the property that child j gets both of their shoes back, and let $P = \{p_1, p_2, \ldots, p_m\}$. Then $N_=(\emptyset)$ is the number of rearrangements of shoes such that no child gets both of their shoes back, and the Inclusion-Exclusion formula says that

$$N_=(\emptyset) = \sum_{J\colon\ \emptyset \subset J \subset P} (-1)^{|J|} N_\geq(J).$$

We must compute these $N_\geq(J)$. Suppose $|J| = n$. For j such that $p_j \in J$, there are 2 ways for Child j to put his own shoes on (the right way, or backwards), so for these n children, there are 2^n ways for them to put their own shows back on. The other $m - n$ children can put on whatever of the remaining shoes they want. There are $2(m - n)$ feet left to put shoes on, and there are $2(m - n)$ shoes to put on them, so there are $(2m - 2n)!$ ways for these other children to put the remaining shoes on. In all, there are $2^n(2m - 2n)!$ ways for the children j, such that $p_j \in J$, to put their own shoes back on and the other children to put on whatever shoes, and so $N_\geq(J) = 2^n(2m - 2n)!$.

Note that $N_\geq(J)$ only depends on $|J|$. Since, for $1 \leq n \leq m$, there are $\binom{m}{n}$ subsets of P of size n, we have that the number of rearrangements of shoes such that no child gets both their shoes back is

$$N_=(\emptyset) = \sum_{n=0}^{m} (-1)^n \binom{m}{n} 2^n (2m - n)!.$$

b. This one is a little painful. One thing I have been very careful to do (and you should to!) is to define properties, sets, etc., very carefully and concretely. Again we will use Inclusion-Exclusion. For $1 \leq j \leq m$, let p_{jL} and p_{jR} be the properties that Child j gets their left shoe back and right shoe back, respectively, and let $P = \{p_{1L}, p_{1R}, \ldots, p_{mL}, p_{mR}\}$. Then we want to compute $N_=(\emptyset)$ (the number of rearrangements where nobody gets either their left or their right shoe back).

Unfortunately, for $J \subset P$, $N_\geq(J)$ doesn't depend only on $|J|$, as it did in part a. Let's define two sets based on J. Let

$$J_1 = \{j : \text{ either } p_{jL} \in J \text{ or } p_{jR} \in J \text{ but not both}\}$$

(that is, the children who we want to ensure get one of their shoes back but we don't care about the other) and

$$J_2 = \{j : \ p_{jL} \in J \text{ and } p_{jR} \in J\}$$

(the children who we want to ensure get both of their shoes back). Note that $|J| = |J_1| + 2\,|J_2|$, because for every child in J_2 both p_{jL} and p_{jR} are in J.

Let $n_1 = |J_1|$ and $n_2 = |J_2|$. For each child in J_1 there are two ways for them to put their own shoe on (either on their right foot or on their left foot); let's leave their other foot bare for the moment. For each child in J_2, there are two ways to put on both of their own shoes (correctly or backwards). Now we have $m - n_1 - n_2$ children with no shoes on, n_1 children with one shoe on, and n_2 children with two shoes on. Then we have

$$2(m - n_1 - n_2) + n_1 = 2m - 2n_2 - n_1$$

bare feet left, and so shoes can be put on them in $(2m - 2n_2 - n_1)!$ ways. Therefore

$$N_{\geq}(J) = 2^{n_1+n_2}(2m - 2n_2 - n_1)!$$

Note that this only depends on the size of n_1 and of n_2. For a given n_1 and n_2, how many sets J of properties are there such that $|J_1| = n_1$ and $|J_2| = n_2$? First, there are $\binom{m}{n_1}$ choices for the set J_1. Next, we can decide for each child in J_1 whether p_{jL} or p_{jR} is in J (that is, whether that child gets their left or right shoe back), and this can be done in 2^{n_1} ways. Finally, we can choose J_2 from among the remaining $m - n_1$ children in $\binom{m-n_1}{n_2}$ ways, so in all, there are

$$\binom{m}{n_1} 2^{n_1} \binom{m - n_1}{n_2}$$

sets J of properties are there such that $|J_1| = n_1$ and $|J_2| = n_2$.

Inclusion-Exclusion, then, gives us

$$\begin{aligned} N_{=}(\emptyset) &= \sum_{J\colon\ \emptyset \subset J \subset P} (-1)^{|J|} N_{\geq}(J) \\ &= \sum_{n_1=0}^{m} \sum_{n_2=0}^{m} (-1)^{n_1+2n_2} \binom{m}{n_1} 2^{n_1} \binom{m - n_1}{n_2} 2^{n_1+n_2} (2m - 2n_2 - n_1)! \end{aligned}$$

ways in which no one ends up with either of their own shoes.

Problem 3

We want to show that

$$\sum_{k=0}^{m} k \cdot S(n+k, k) = S(m+n+1, m).$$

The righthand side of this equation is the number of ways to partition $\{1, 2, \ldots, m+n+1\}$ into m parts. We will count this number in another way, and show that that gives us the lefthand side. (Note: some people proved this by induction using Theorem 3.1, and that worked out well too).

We can construct all partitions of $\{1, 2, \ldots, m+n+1\}$ into m parts, as follows. First, we choose an integer k and say that $\{n+k+2\}, \{n+k+3\}, \ldots, \{n+m+1\}$ will each be parts of the partition, but that $n+k+1$ will not be alone in its piece. Note that we must have $k \geq 0$, because if k were negative, then this would already give us $m-k > m$ parts. Also, we must have $k \leq m$ or this doesn't make any sense. This gives us

$$(n+m+1) - (n+k+2) + 1 = m - k$$

parts already, and now we must break up what is left, $\{1, 2, \ldots, n+k+1\}$, into the remaining k parts (such that $n+k+1$ is not alone in its part). To do this, partition $\{1, 2, \ldots, n+k\}$ into k parts (in $S(n+k, k)$ possible ways) and then add $n+k+1$ to any one of these k parts (in k possible ways). We now have a partition of $\{1, 2, \ldots, m+n+1\}$ into m parts. In all, this gives us

$$\sum_{k=0}^{m} k \cdot S(n+k, k)$$

ways to construct the partition.

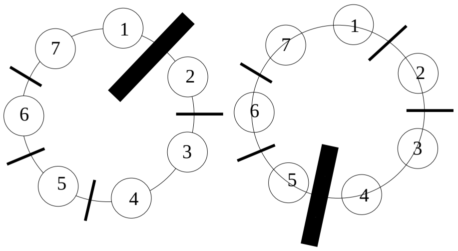
To make sure we are not overcounting or undercounting, we should make sure that this process is reversible. This will show that we have counted every partition and haven't counted any partition more than once. Given a partition of $\{1, 2, \ldots, m+n+1\}$ into m parts, there is a unique integer k such that $\{n+k+2\}, \{n+k+3\}, \ldots, \{n+m+1\}$ are each parts of the partition, but that $n+k+1$ is not alone in its piece. Throw out $\{n+k+2\}, \{n+k+3\}, \ldots, \{n+m+1\}$ and we have a partition of $\{1, 2, \ldots, n+k+1\}$ into k parts such that $n+k+1$ is not alone in its part. Since $n+k+1$ is not alone, we can throw it out and have a partition of $\{1, 2, \ldots, n+k\}$ into still k parts. Therefore this count is correct, and

$$\sum_{k=0}^{m} k \cdot S(n+k, k) = S(m+n+1, m).$$

Problem 4

I'd like to point out a couple of things about my solution to this problem as (hopefully!) a model of what a careful proof should be. One thing is that careful writing helps us do correct mathematics. This problem was easy to get wrong if you weren't careful. Another point is that pictures and examples can be very helpful (though not a substitute for careful wording).

Let's count the number of ways to make one "bold cut" (i.e., distinguished from the rest of the cuts) and then $l-1$ other cuts, so that we have m_i pieces of size i. We will have overcounted what we want to find by a factor of l, because given a cutting of the necklace into l pieces of the appropriate sizes, we could choose any of the l cuts to be the bold cut (for example, the two pictures below have a different bold cut, but they correspond to the same cutting of the necklace).



First let's make the bold cut, and there are n places to make it (between any pair of adjacent objects). After this cut, our necklace can now be strung out in a line. How many ways are there to make the $l-1$ other cuts? Let's take an alphabet of m_1 "a_1"s, m_2 "a_2"s, …, m_k "a_k"s. Any "word" formed using all of the letters corresponds to a way to cut the line up, e.g, the word "$a_1a_2a_1a_1a_2$" means cut it so that the first piece has 1 object, the second piece has 2 objects, the third piece has 1 objects, the fourth piece has 1 object, and the last piece has 2 objects (as in the left hand picture above, counting clockwise from the bold cut). By Theorem 1.5 from the textbook the number of words using all of the letters is

$$\frac{(m_1+m_2+\cdots+m_k)!}{m_1!m_2!\cdots m_k!}=\frac{l!}{m_1!m_2!\cdots m_k!}.$$

Putting it all together, the number of ways to make the first bold cut was n, the number of ways to finish cutting the necklace was $\frac{l!}{m_1!m_2!\cdots m_k!}$, and we have overcounted by a factor of l, so the total number of ways of cutting up the necklace so that there are m_i pieces of size i is

$$\frac{n \cdot l!}{l \cdot m_1! m_2! \cdots m_k!} = \frac{n(l-1)!}{m_1! m_2! \cdots m_k!}.$$

Problem 5

Note: we're counting equivalence classes under a "symmetry." This will be the chief subject of Chapter 8 (Polya Theory), which we will cover later.

a. As a start, we count the number of ways to place 1 cranberry and n pieces of popcorn in a line. This is $\binom{n+1}{1} = n+1$. But we have overcounted, because we can flip the string around. Most strings, in fact, we have counted exactly twice (the string and its flipped version). The only strings which we counted only once are the ones that are symmetric (if you flip them, you get the same string back).

If n is even, then there is 1 symmetric string: $\frac{n}{2}$ popcorns, followed by the cranberry, followed by $\frac{n}{2}$ popcorns. Then $(n+1) - 1 = n$ is twice the number of non-symmetric strings, and so $\frac{n}{2}$ is the total number of non-symmetric strings. Therefore the total number of strings is the number of non-symmetric plus the number of symmetric, which is $\frac{n}{2} + 1$.

Similarly, if n is odd, there are no symmetric strings (if the cranberry goes between a popcorns on the left and b popcorns on the right, then a has to equal b for the string to be symmetric, but that would mean that $n = a + b = 2a$ is even, not odd). Therefore the total number of strings is

$$\frac{1}{2}\left[n + 1 - 0\right] + 0 = \frac{n+1}{2}.$$

b. We use the same reasoning as part a. There are $\binom{n+2}{2}$ ways to place 2 cranberries and n popcorns in a line, and this counts all of the non-symmetric strings twice (and the symmetric strings only once). The symmetric strings must look like a popcorns, one cranberry, b popcorns, one cranberry, followed by a popcorns, where a and b are nonnegative integers, and $2a + b = n$.

If n is even, then there are $\frac{n}{2}$ choices for a (and then b must be $n - 2a$), so there are $\frac{n}{2}$ symmetric strings. Therefore, as in part a, the total number of strings is

$$\frac{1}{2}\left[\binom{n+2}{2} - \frac{n}{2}\right] + \frac{n}{2} = \frac{n^2 + 4n + 4}{4}.$$

If n is odd, there are $\frac{n-1}{2}$ symmetric strings, and so the total number of strings is

$$\frac{1}{2}\left[\binom{n+2}{2}-\frac{n-1}{2}\right]+\frac{n-1}{2}=\frac{n^2+4n+3}{4}.$$

c. As before, there are $\binom{n+k}{k}$ ways to place k cranberries and n popcorns in a line, and this counts the non-symmetric strings twice. There are $n+k$ total positions on the string that either popcorns or cranberries will go in. Suppose we have a symmetric string. Then if there are l cranberries in the first $\frac{n+k}{2}$ positions, then there must also be l cranberries in the last $\frac{n+k}{2}$ positions, placed symmetrically to the first l (we are using the fact the n and k are assumed to both be even). Therefore $l+l=k$ and $l=\frac{k}{2}$. Conversely, for any placement of $\frac{k}{2}$ cranberries in the first $\frac{n+k}{2}$ positions, there is exactly one way to create a symmetric string (by placing the last $\frac{k}{2}$ cranberries in the last $\frac{n+k}{2}$ positions in symmetric order). Therefore there are

$$\binom{\frac{n+k}{2}}{\frac{k}{2}}$$

symmetric strings, and so the total number of strings is

$$\frac{1}{2}\left[\binom{n+k}{k}-\binom{\frac{n+k}{2}}{\frac{k}{2}}\right]+\binom{\frac{n+k}{2}}{\frac{k}{2}}=\frac{1}{2}\binom{n+k}{k}+\frac{1}{2}\binom{\frac{n+k}{2}}{\frac{k}{2}}.$$