Computing characteristic classes of foliations & flat bundles

Last week, we saw that for finite dim'l Lie group G w/ maximal cpt K

Chern-Weil theory gives us vertical maps:

$$\begin{tikzcd}
H^*(B\bar{G}) & H^*(BG^\delta) \arrow[l] & H^*(BG) \cong H^*(BK) \arrow[l] \\
H^*(\mathfrak{g}) \arrow[u] & H^*(\mathfrak{g},\mathfrak{k}) \arrow[l] \arrow[u] & I(K) \arrow[l] \arrow[u]
\end{tikzcd}$$

Low dimensional example: $G = SL_2\mathbb{R}$, $K = SO(2)$

$\mathfrak{g} = \mathfrak{sl}_2\mathbb{R}$ has basis $X_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Let ϕ_i be dual basis in $\mathfrak{sl}_2\mathbb{R}^*$.

Then:

$$\left.\begin{aligned} d\phi_0 &= -\phi_1 \wedge \phi_2 \\ d\phi_1 &= -2\phi_0 \wedge \phi_1 \\ d\phi_2 &= 2\phi_0 \wedge \phi_2 \end{aligned}\right\} \text{ follows from } [X_0, X_1] = 2X_1 \text{ etc } \;\&\; d\phi(X,Y) := -\phi([X,Y])$$

We did this computation in Godbillon-Vey intro lecture

compute: $H^k(\mathfrak{sl}_2\mathbb{R}) = \begin{cases} \mathbb{R} & k = 0, 3 \\ 0 & \text{else} \end{cases}$

Relative cohomology is cohomology of $C^*(\mathfrak{g},\mathfrak{k}) := \left\{ \phi \in C^*(\mathfrak{g}) : i(X)\phi = i(X)d\phi = 0 \text{ for all } X \in \mathfrak{k} \right\}$

(can also equivalently compute cohomology of $Ad(K)$-invariant forms satisfying $i(X)\phi = 0$ for $X \in \mathfrak{k}$.)

$\mathfrak{so}_2$ has 1-dim'l basis $Y = -X_1 + X_2$.

$i(Y)\phi_0 = 0$

$i(Y)\phi_1 = -1$

$i(Y)\phi_2 = 1$

possible elements of $C^*(\mathfrak{g},\mathfrak{k})$ so far: ϕ_0, $\phi_1 + \phi_2$, $\phi_0 \wedge (\phi_1 + \phi_2)$ ...

$i(Y)\,d\phi_0 = i(Y)(-\phi_1 \wedge \phi_2) = -\phi_1(Y) \wedge \phi_2 + \phi_1 \wedge \phi_2(Y) = \phi_2 + \phi_1 \neq 0.$

$i(Y)\,d(\phi_1 + \phi_2)$ also nonzero (check!)

$i(Y)\,d(\phi_0 \wedge (\phi_1 + \phi_2)) = 0$

conclude $H^k(\mathfrak{sl}_2\mathbb{R}, \mathfrak{so}_2) = \begin{cases} 0 & k \neq 2, 0 \\ \mathbb{R} & k = 2, 0 \end{cases}$

We have:

$$\begin{tikzcd}
H^*(B\overline{SL_2\mathbb{R}}) & H^*(BSL_2\mathbb{R}^\delta) \arrow[l] & H^*(BSL_2\mathbb{R}) \cong H^*\mathbb{C}P^\infty = \mathbb{R}[\varepsilon] \text{ Euler class} \arrow[l] \\
\text{G-V class} \quad \phi_0 \wedge \phi_1 \wedge \phi_2 \text{ in dimension 3} \arrow[u] & \phi_0 \wedge (\phi_1 + \phi_2) \text{ in dimension 2} \arrow[u, hook]
\end{tikzcd}$$

Brooks-Goldman: GV is not a continuous class for $SL_2\mathbb{R}$ foliation.

If instead, $G = \mathrm{Diff}_0(M^q)$, we have a less clear picture:

$$\begin{tikzcd}
H^*(B\bar{G}) & H^*(BG^\delta) \arrow[l] & H^*(BG) \arrow[l] \\
H^*_{GF}(M) \arrow[u] & \begin{array}{c}\text{some classes can be obtained from}\\ H^*(a_q, O_q)\otimes H^*(BG)\end{array} \arrow[u, "\text{not injective}"'] & ?? \arrow[u]
\end{tikzcd}$$

understood in a few cases where we know the homotopy type of G.

In the $M = S^1$ case, we understand the picture (but this is because $S^1 \subset \mathrm{Diff}_0 S^1$ behaves like a "maximal compact subgroup")

$$\begin{tikzcd}
H^*(B\overline{\mathrm{Diff}_0 S^1}) & H^*(B\mathrm{Diff}_0 {S^1}^\delta) \arrow[l] & H^*(B\mathrm{Diff}_0 S^1) = H^*(\mathbb{C}P^\infty) \arrow[l] \\
H^*_{GF}(S^1) \arrow[u] & \begin{array}{c} H^*(a_1) \cong \mathbb{R}[\alpha] \\ \otimes \\ H^*(\mathbb{C}P^\infty) \cong \mathbb{R}[\varepsilon] \end{array} \arrow[u, "\text{kernel is } \alpha\varepsilon"'] &
\end{tikzcd}$$

$H^*(B\mathrm{Diff}_0 S^1) \cong \mathbb{R}[\varepsilon]$ Euler class.

α: degree 3

$H^*_{GF}(S^1) \cong \mathbb{R}[\alpha] \otimes \Lambda(\beta)$

α: "integrating GV along fiber" degree 2

β: "pull back GV from flat bundle" degree 3.

Goal for today: Explain objects & maps & how to compute H^*, starting with $H^*(a_q, O_q)$ to finish work in last talk.

<u>Last time</u>: we saw 3 different (but related) objects

① Differential graded algebra $WO(q) := \Lambda(u_1, \ldots u_k) \otimes \underline{P_q}(c_1, \ldots c_q)$

u_i: i odd, $i \leq q$ $\quad$ P_q: polynomials truncated at deg $2q$.

(Motivated by Chern-Weil theory & Bott vanishing)

② Lie algebra of formal vector fields on $\mathbb{R}^q$.
(Sam called this $\mathcal{L}_q$, I'm going to call it a_q, following Bott & Morita)

$H^*(a_q)$ = cohomology of complex of continuous alternating forms $a_q \times \ldots \times a_q \to \mathbb{R}$

Possible, but tedious, to compute explicitly.

Computing $H^*(a_q)$: Elements of a_q are $\sum a_i \frac{\partial}{\partial x_i}$, a_i formal power series in x's.

define 1-forms: $\theta^i\left(\sum a_k \frac{\partial}{\partial x_k}\right) = a_i(0)$

$\theta^i_j\left(\quad " \quad \right) = \frac{-\partial}{\partial x_j}(a_i)\big|_0$

$\theta^i_{jl}\left(\quad " \quad \right) = \frac{-\partial^2}{\partial x_j \partial x_l}(a_i)\big|_0$ etc.

(sign choice arranged so that $\mathcal{L}_{\left(\frac{\partial}{\partial x_j}\right)} \theta^i = \theta^i_j$)

Easy to check that: • $d\theta^i = -\sum_j \theta^i_j \wedge \theta^j$ ⊛

• These 1-forms span

Define 2-forms "curvature" $R^i_j := d\theta^i_j + \sum \theta^i_k \wedge \theta^k_j$

(Why "curvature"? if ω is connection then $\Omega = d\omega + \omega\wedge\omega$. Componentwise, thinking of ω as a matrix of 1-forms, $\Omega^i_j = d\omega^i_j + \sum \omega^i_k \wedge \omega^k_j$)

Can prove "Bianchi identities"

• $\sum_j R^i_j \wedge \theta^j = 0$

• $dR^i_j = \ldots\ldots$ (see Bott)

which can all be derived from ⊛ by taking appropriate Lie derivatives.

These imply that the subalgebra generated by R^i_j's & θ^i_j's is finite dim'l & <u>closed under d</u>.

<u>Example</u>: $q=1$. Subalgebra generated by R^1_1 and θ^1_1

$d\theta^1_1 = R^1_1$ by definition

$dR^1_1 = 0$

$d(R^1_1 \wedge \theta^1_1) = d(d\theta^1_1 \wedge \theta^1_1) = 0$

so $H^*(\text{subalgebra}) \cong \begin{cases} \mathbb{R} & \text{degree } 0, 3 \\ 0 & \text{else} \end{cases}$ we'll see that $R^1_1 \wedge \theta^1_1$ is GV class.

$H^*(a_q; O_q)$ = cohomology of formal VF's invariant under O_q adjoint action, similar. (in $q=1$ case, exactly the same.)

<u>Theorem</u> (Gelfand - Fuks) Highly nontrivial proof.

$H^*(a_q, O_q) \cong H^*(WO_q)$ $\qquad H^*(a_q) \cong H^*(W_q) := \Lambda(u_1 \ldots u_q) \otimes P_q[c_1 \ldots c_q]$

$c_i(R) \longleftrightarrow c_i$

thinking of "R" as $\{R^i_j\}_{ij}$

complicated $\longleftrightarrow u_i$ $\qquad$ $q=1$ case is easy: $c_1 \mapsto R^1_1$, $u_1 \mapsto \theta^1_1$

3 Canonical forms on jet bundles:

<u>Q</u>: How does $H^*(a_q, O_q)$ give us characteristic classes of foliations?

<u>A</u>: Jet Bundles. Let $G^k(q)$ = k-jets at 0 of germs of diffeos of $\mathbb{R}^q$ fixing 0.

Given codim-q foliation $\mathcal{F}$ on M, define $J^k(\mathcal{F})$ = k-jets of loc. submersions $M \to \mathbb{R}^q$ defining $\mathcal{F}$ (kernel is $T\mathcal{F}$)

$$\to J^k(\mathcal{F}) \to \dots \to J^2(\mathcal{F}) \to \underset{\substack{\text{normal bundle} \\ \text{to } \mathcal{F}}}{\underset{\|}{J^1(\mathcal{F})}} \to M$$

$J^k(\mathcal{F}) \to M$ is a principal $G^k(q)$ bundle, $G^k(q)/G^{k-1}(q) \sim *$
$G^1(q)/O(q) \sim *$

<u>Claim</u>: $H^*(a_q) \to H^*(J^\infty(\mathcal{F}))$ (induced by formal v.f. at 0 on $\mathbb{R}^q$ pulled back by loc. submersion) has image, generated by the canonical / tautological 1-forms

and $H^*(a_q, O_q) \to H^*(J^\infty(\mathcal{F})/O(q)) \cong H^*(J^1(\mathcal{F})/O(q)) \cong H^*(M)$
has image the $O(q)$-invariant tautological forms,
& this gives <u>characteristic classes of $\mathcal{F}$</u>.

<u>Remark</u>: If we are given a trivialization of the normal bundle $\nu(\mathcal{F})$, i.e. a section $M \to J^1(\mathcal{F})$, then we don't need to work with $H^*(a_q, O_q)$. Instead:

$$H^*(a_q) \to H^*(J^\infty(\mathcal{F})) \cong H^*(J^1(\mathcal{F})) \xrightarrow{s^*} H^*(M).$$

Description of canonical 1-forms is in Sam's notes from last time. The correspondence with $H^*(a_q)$ is
via $\theta^i \longleftrightarrow \omega_i$
$\theta^i_j \longleftrightarrow \theta_{ij}$

We have $d\theta^i = -\sum_{j=1}^{n} \theta^i_j \wedge \theta^j$, etc.

(continued on next page →)

Given characteristic class of ~~of~~ codim-q foliation and manifold N^q can produce characteristic classes of foliated N^q bundles by integrating along the fiber. $N^q \to M \downarrow B$ flat bundle $\to \mathcal{F}$ foliation transverse to fibers.

$$\alpha \in H^k(M;\mathbb{R}) \xrightarrow{\int_N} H^{k-q}(B;\mathbb{R})$$

So we get a map $H^*(a_q, O_q) \longrightarrow H^* BDiff(N^\delta)$

Can also first take cup product with a characteristic class of N-bundles, so get map $H^*(a_q, O_q) \otimes H^*(BDiff(N)) \longrightarrow H^* BDiff(N)^\delta$

S¹ example:

$$H^*(BDiff_0 S^1) \leftarrow H^*(BDiff_0^\delta(S^1)) \longleftarrow H^*(BDiff_0 S^1) = H^*(\mathbb{C}P^\infty)$$

$H^*_{GF}(S^1)$ $= \mathbb{R}[\alpha] \otimes \Lambda(\beta)$ ← pull back GV via section; ↑ integrate GV along fiber

$H^*(a_1) \cong \mathbb{R}[\alpha]$ $\otimes$ $H^*(\mathbb{C}P^\infty) \cong \mathbb{R}[\varepsilon]$

Godbillon-Vey class in H^3 integrated along fiber to give class in $BDiff_0^\delta(S^1)$

Explicit cocycle representatives known: (we'll prove later)

$$\alpha\left(f\tfrac{\partial}{\partial t}, g\tfrac{\partial}{\partial t}\right) = \int_{S^1} f'g'' - f''g' = 2\int_{S^1} f'g'' \quad (\text{~~integration by parts~~ stokes})$$

$$\beta\left(f\tfrac{\partial}{\partial t}, g\tfrac{\partial}{\partial t}, h\tfrac{\partial}{\partial t}\right) = \int_{S^1} \begin{vmatrix} f & f' & f'' \\ g & g' & g'' \\ h & h' & h'' \end{vmatrix} dt$$

Thurston's cocycle formula for (integral along fiber of) gv in $H^* BDiff_0^\delta(S^1)$:

$$c(f,g) = \int_{S^1} \log g' \; D\log (f_g)' \, d\theta$$

Proof: show that, for f, g in $PSL_2\mathbb{R}$, you get gv. This computation has been done explicitly by Brooks in appendix to paper of Bott.

Some remarks on computations of $H^*_{GF}(M)$

Method #1 (Original GF method)

C^q = continuous $\mathbb{R}$-multilinear maps $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathbb{R}$.

Define filtration by "dimension of support": α has degree $\leq k$ if, whenever $\alpha(X_1, X_2 \ldots X_q) \neq 0$ $\exists$ k points in M s.t. each X_i not $\equiv 0$ in nbd of at least 1 point.

(e.g. $\leq q$ is always satisfied, smaller if supports of X_i intersect)

filtration preserved by d.

Further filtration taking into account multiplicites of 0's of vanishing VF's, differential is $C^q_{k,m} \to C^{q+1}_{k,m+1}$.

Use this info to define spectral sequence: E_0 page is $E_0^{q-m,m} = C^q_{1,m} / C^q_{1,m-1}$

E_1, E_2 pages have a reasonable description involving $H^*(M)$ and Lie algebra of formal VF's.

E_∞ computes H^*_{GF} (or at least of a certain "diagonal subcomplex"...)

Not really practical except: Thm: if M is paralellizable (e.g S^1) differentials are 0 starting at E_2 page.

Method #2 (Haefliger, Bott-Segal, Trauber ...)

Define bundle $\begin{array}{c} E \\ \downarrow \\ M^n \end{array}$ where fiber F has $H^*_{dR}(F) \cong H^*(a_n)$ and

(actually, also some subtlety involving $O(n)$ action)

$H^*(\Gamma E) \cong H^*_{GF}(M)$.

space of continuous section

" a_n gives the ~~...~~ GF cohomology at a point"

In the case $M = S^1$, $H^*(a_1) = \begin{cases} \mathbb{R}, & k = 0, 3 \\ 0 & \text{else} \end{cases}$ so we take $F = S^3$.

Bundles ~~...~~ over S^1 are trivial, so $E = S^1 \times S^3$

$\Gamma E = \mathrm{Map}(S^1, S^3)$ and $H^*(\Gamma E) = \underset{\deg 2}{\mathbb{R}[\alpha]} \otimes \underset{3}{\Lambda(\beta)}$.

Cocycle Representatives

Proof that $H^2_{GF}(S^1)$ generator is represented by cocycle

$$c\left(f\tfrac{\partial}{\partial t},\ g\tfrac{\partial}{\partial t}\right) = \int_{S^1} f' g''\, dt. \qquad \text{(following Khesin - Wendt):}$$

1. Work with complexification of Lie algebra of vector fields on S^1

   $f(t)\frac{\partial}{\partial t}$ in $\mathcal{X}(S^1)\otimes\mathbb{C}$ has fourier series $f(t) = \sum f_n e^{int}$.

   Continuous cocycles are completely determined by values on basis
   $\{ L_n := i e^{int} \frac{\partial}{\partial t} \}$.

   Note that $[L_n, L_m] = (m-n) L_{n+m}$ ⊛

   The cocycle identity $dc = 0$ gives

   $$c([L_0, L_m], L_n) + c(L_m, [L_0, L_n]) = \underbrace{c(L_0, [L_m, L_n])}_{=\partial\alpha \text{ where } \alpha(L_m) := c(L_0, L_m) \text{ i.e. } i_{L_0}(c)}.$$

We'll show that the space of multilinear c's satisfying the cocycle condition above is at most 2 dimensional (and one of the candidates will be exact.)

⊛ $\Rightarrow m\, c(L_m, L_n) + n\, c(L_m, L_n) = 0$, so $c(L_m, L_n) = 0$ unless $m+n=0$.

So we just need to understand $c(L_n, L_{-n})$.

Look at cocycle identity for L_1, L_n, L_{-1-n} to get recursive formula

$$(n+1)\, c(L_{n+1}, L_{-(n+1)}) + (n+2)\, c(L_n, L_{-n}) - (2n+1)\, c(L_1, L_{-1}) = 0.$$

This implies space is ≤ 2 dimensional. Explicitly, we can solve for ~~basis~~ linearly indep. elements

$$c(L_n, L_{-n}) = n^3$$

$$c(L_n, L_{-n}) = n \leftarrow \text{but this one is } d\beta \text{ where } \beta(L_n) = \begin{cases} -\frac{1}{2} & n=0 \\ 0 & \text{else.} \end{cases}$$

You can check that the n^3 cocycle is nontrivial (see Khesin - Wendt) (gives new proof that $H^2 \neq 0$, not hard)

but we already know $H^2_{GF}(S^1) \neq 0$.

Now check that it agrees with our original cocycle by looking at derivatives of fourier series