Introduction to Gelfand-Fuks cohomology

§ Motivation

G connected lie group.

Chern-Weil theory : $\begin{matrix} P \\ \downarrow \pi \\ M \end{matrix}$ principle G-bundle

A connection on $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$ is 1-form $\omega \in \Omega^1(P; \mathfrak{g})$ & curvature $\Omega \in \Omega^2(P;\mathfrak{g})$

$\Omega^k \in \Omega^{2k}(P; \mathfrak{g}^{\otimes k})$. If we have an invariant poly $f \in I^k(G)$:

$$f : \underbrace{\mathfrak{g} \times - \times \mathfrak{g}}_{k\text{-times}} \longrightarrow \mathbb{R}$$

invariant under adjoint action of G.

$f(\Omega^k) \in \Omega^{2k}(P)$. $\rightsquigarrow$ It comes from closed form on the base

Def, Weil Algebra $W(\mathfrak{g}) = \Lambda^* \mathfrak{g}^* \otimes S^* \mathfrak{g}^*$ (de Rham Model of the total space $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$).

$$\begin{tikzcd} W(\mathfrak{g}) \arrow[r, "\tilde{\omega}"] & \Omega^*(P) \\ I(G) \arrow[u, "i"] \arrow[r, "\bar{\omega}"] & \Omega^*(M) \arrow[u, "\pi^*"'] \end{tikzcd}$$

Connection defines a linear map $\omega : \mathfrak{g}^* \longrightarrow \Omega^1(P)$

$$\begin{tikzcd} \omega : \Lambda^* \mathfrak{g}^* \arrow[r] \arrow[d, phantom, "\wr"] & \Omega^*(P) \arrow[r, squiggly, "\text{flatness}"] & \omega^* : H^*(\Lambda \mathfrak{g}^*, d) \arrow[r] & H^*(\Omega^*(P)) \arrow[d, phantom, "\wr"] \\ \Omega^*(G)^G & & & H^*(P) \end{tikzcd}$$

$d: \Lambda^k \mathfrak{g}^* \longrightarrow \Lambda^{k+1} \mathfrak{g}^*$ $\qquad d\varphi(X_1, -, X_{k+1}) = \sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots X)$

this gives the lie alg cohomology of $\mathfrak{g}$.

if $K \subset G$ is sub lie group & $k \subset \mathfrak{g}$ its sub lie alg

__Def__, Relative Complex $C^*(\mathfrak{g}, k) = \{ \varphi \in C^*(\mathfrak{g}) : i_X \varphi = \mathcal{L}_X \varphi = 0$ for any $X \in k \}$.

__flat product G-bundle__ $\quad M \times G = P \xrightarrow{\pi} M \qquad \omega^*: H^*(\mathfrak{g}) \longrightarrow H^*(P; \mathbb{R}) \xrightarrow{\text{section}} H^*(M; \mathbb{R})$

__flat G-bundle__, let $K \subset G$ be maximal compact sub group.

$$P \longrightarrow P/K \longrightarrow M \quad , \quad \begin{tikzcd} G/K \arrow[r] & P/K \arrow[r] & M \arrow[l, bend right, hook, "s"'] \end{tikzcd}$$

G/K is * contractible

$$\begin{tikzcd} C^*(\mathfrak{g}) = \Lambda^* \mathfrak{g}^* \arrow[r, "\omega"] & \Omega^*(P) \\ C^*(\mathfrak{g}, k) \arrow[u] \arrow[r, "\omega"] & \Omega^*(P/K) \arrow[u] \end{tikzcd}$$

$$\omega: H^*(\mathfrak{g}, k) \longrightarrow H^*(M, \mathbb{R})$$

__Universal case__:

$$\overline{BG} \longrightarrow BG^{\delta} \longrightarrow BG$$

classifies flat G-bdls

classifies product G-bdls

$$\begin{tikzcd}
H^*(\mathfrak{g}) \arrow[r] & H^*(\overline{BG}; \mathbb{R}) \\
H^*(\mathfrak{g},k) \arrow[u] \arrow[r] & H^*(BG^\delta; \mathbb{R}) \arrow[u] \\
I(K) \arrow[u] \arrow[r] & H^*(BK; \mathbb{R}) = H^*(BG; \mathbb{R}) \arrow[u]
\end{tikzcd}$$

Rk: Thm (Harish-Chandra, Koszul, –) $H^*(\mathfrak{g},k) \longrightarrow H^*(BG^\delta; \mathbb{R})$ is inj

Thm (Van Est) The image of the map is continuous cohomology of G^δ.

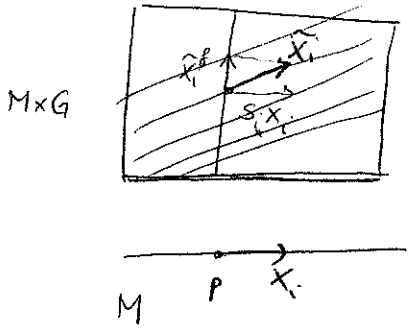
Gelfand–Fuks point of view;

$$\begin{tikzcd}
P = M \times G \arrow[r] & M \arrow[l, bend right, "s"']
\end{tikzcd}$$

Goal: to define $H^*(\mathfrak{g}) \xrightarrow{s^* \circ \omega^*} H^*(M)$

Let $\varphi \in C^k(\mathfrak{g})$

Suppose $X_1, \ldots, X_k \in T_pM$



$$s^* \circ \omega^* (\varphi)(X_1, \ldots, X_k)$$
$$= \omega(\varphi)(s_* X_1, \ldots, s_* X_k)$$
$$= (-1)^k \varphi(\tilde{X}_1^\delta, \ldots, \tilde{X}_k^\delta)$$

Secondary characteristic classes (GV point of view)

$\mathcal{F}$ foliation on M.

ν normal bdle of F.

Bott Connection.

$$D : TM \times \Gamma(V) \longrightarrow \Gamma(V)$$

for $X \in \mathcal{X}_{\mathcal{F}}$ (tangent to F) & $V \in \Gamma(V)$

choose $\hat{V}$ such that $\pi : T(M) \longrightarrow TM/\tau(F)$

$$\hat{V} \longrightarrow V$$

$$D'_X(V) = \pi([X, \hat{V}])$$

$$X \in TM = T\mathcal{F} \oplus \nu\mathcal{F}$$
$$X' + X''$$

$$D_X(V) = D'_X(V) + D''_{X''}(V)$$

any other connection.

$$P \in I^*(Gl_q(\mathbb{R})) \quad \text{if} \quad [P(\Omega)] \in H^{2k}(M; \mathbb{R})$$

Say we have two connections D^0, D^1, $P \in I^k(Gl_q(\mathbb{R}))$

$$[P(\Omega^0)] = [P(\Omega^1)] \qquad D^t = tD^1 + (1-t)D^0$$

$$\tau P(\Omega^1, \Omega^0) = k \int_0^1 P(D^1 - D^0, \Omega^t, \Omega^t, \ldots \Omega^t)\, dt$$

$$d\tau P(\Omega^1, \Omega^0) = -P(\Omega^0) + P(\Omega^1)$$

if $k > q$ $\quad P(\Omega^1) = 0$ for Bott Connection D^1

let D^0 be a Riemannian connection i.e $X \cdot \langle V, W \rangle = \langle D^0_X V, W \rangle + \langle V, D_X W \rangle$

$\Omega_{X,Y} \in O(q) = \{A \in gL(\mathbb{R}^q) : A + A^t = 0\}$. $\quad P(\Omega) = 0$ for k odd.

let $P \in I^k(Gl_q(\mathbb{R}))$ k odd $k > q$.

$$d\, TP(\Omega^1, \Omega^0) = 0 .$$

Thm $[TP(\Omega^1, \Omega^0)] \in H^{2k-1}(M, \mathbb{R})$ is independent of choices

is invariant of foliation.

Def. Graded Differential Algebra

$$WO_q = \Lambda(u_1, u_3, \ldots, u_{2[\frac{q}{2}]-1}) \otimes P_q(c_1, \ldots, c_q)$$

Where u_k's, c_k's are generators deg $u_k = 2k-1$ deg $c_k = 2k$

$du_k = c_k$, $dc_k = 0$, P_q is poly alg mod elements of total deg $> 2q$.

GV class: $\phi : WO_q \longrightarrow \underset{\text{deRham}}{C^*(M)}$

$$\phi(c_k) = P_k(\underset{\uparrow}{\Omega^1}) \qquad P_k \text{ is } k\text{-sym polynomial}$$

Bott Connection

$$\phi(u_k) = TP_k(\Omega^1, \Omega^0)$$

induces $\rightsquigarrow$ $H^*(WO_q) \longrightarrow H^*(M; \mathbb{R})$ depends only on Foliation

Ex: $u_1 c_1^q$ in WO_q $d(u_1 c_1^q) = c_1^{q+1} = 0$, $\phi(u_1 c_1^q) \in H^{2q+1}(M; \mathbb{R})$

$$\begin{tikzcd}
H_{GF}(M) \arrow[r] & H^*(\overline{B\mathrm{Diff}(M)}) \\
? \arrow[u] \arrow[r] & H^*(B\mathrm{Diff}^{\delta}(M)) \arrow[u] \\
? \arrow[u] \arrow[r] & H^*(B\mathrm{Diff}(M)) \arrow[u]
\end{tikzcd}$$

Def. $\mathcal{X}_F$ denote lie alg of vector fields on F with C^∞-top.

let $C^p(\mathcal{X}_F)$ denote the space of continuous, alternating p-linear from on $\mathcal{X}_F$ with values in $\mathbb{R}$)

$$d: C^p(\mathcal{X}_F) \longrightarrow C^{p+1}(\mathcal{X}_F)$$

$$dc(V_0, \dots, V_p) = \sum_{i<j} (-1)^{i+j} c([V_i, V_j], V_0, V_1, \dots \hat{V}_i, \dots \hat{V}_j, \dots, V_p)$$

$H^*(\mathcal{X}_F)$ is GF-Cohomology of F.

Thm (GF). $H^*_{GF}(M)$ is finite dimensional. for every manifold.

Def. $A \subset M$ is closed subset. Let $\mathcal{X}_{M,A}$ be vector fields V whose infinite $J^\infty(V) \equiv 0$ on A. Set $\mathcal{X}_A = \mathcal{X}_F / \mathcal{X}_{F,A}$.

$$\mathcal{X}_0 = \mathcal{X}_{\mathbb{R}^q} / \mathcal{X}_{\mathbb{R}^q,0} \qquad \text{Lie algebra formal vector field.}$$

$V \in \mathcal{X}_0 \rightsquigarrow V = \sum f_i(x_1, \dots x_q)\, \partial/\partial x_i$ where f_i's are formal powers series.

The usual action O_q on $\mathbb{R}^q$ induces an action on $C^*(\mathcal{X}_0)$

and embeds $O_q \subset \mathcal{X}_0$. $C^*(\mathcal{X}_0, O_q) \subset C^*(\mathcal{X}_0)$

Def.

$$W_q = \Lambda(u_1, \dots, u_q) \otimes P_q(c_1, \dots, c_q)$$

$$WO_q = \Lambda(u_1, u_3, \dots, u_{2[\frac{q}{2}]-1}) \otimes P_q(c_1, \dots, c_q)$$

$$du_i = c_i \qquad dc_i = 0.$$

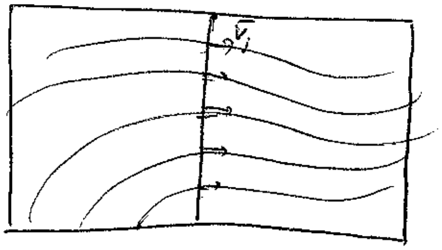
Thm (GF): There exist maps:

$$W_q \longrightarrow C^*(\mathcal{X}_0), \quad WO_q \longrightarrow C^*(\mathcal{X}_0, O_q)$$

that induces iso on cohomology.

if $\dim(F) = q$ and $\mathcal{F}$ codim q foliation on $M \times F$:

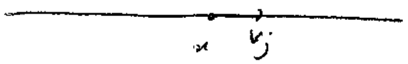
$$\phi_{\mathcal{F}} : \underset{\overset{\cup}{c}}{C^*(\mathcal{X}_F)} \longrightarrow \underset{\text{deRham}}{C^*(M)}$$



let $v_1, \dots, v_p \in T_x(M)$

let $\tilde{v}_i^x$ be the proj $\tilde{v}_i$ onto F

$$\phi_{\mathcal{F}}(c)(v_1, \dots, v_p) := c(\tilde{v}_1^x, \dots, \tilde{v}_p^x)$$



Ex: $d\phi_{\mathcal{F}}(c) = \phi_{\mathcal{F}}(dc)$

We can think of $\overline{\Gamma_q}$-structure $\bar{H}$ on M as germ of foliation of codim q on $M\times\mathbb{R}^q$ transverse to $\mathbb{R}^q$-factor along $M\times\{0\}$.

$$C^*(\mathcal{X}_{\mathbb{R}^q}) \longrightarrow C^*_{\text{deRham}}(M)$$

$$C^*(\mathcal{X}_0) = \ker\left(C^*(\mathcal{X}_{\mathbb{R}^q}) \longrightarrow C^*(\mathcal{X}_{\mathbb{R}^q,\{0\}})\right)$$

$$\phi_{\bar{H}} : C^*(\mathcal{X}_0) \longrightarrow C^*_{\text{deRham}}(M)$$

$$\phi_{\bar{H}} : H^*(\mathcal{X}_0) \longrightarrow H^*(M)$$

induces $$H^*(\mathcal{X}_0) \longrightarrow H^*(B\overline{\Gamma_q})$$

$$\phi_{\mathcal{H}} : C^*(\mathcal{X}_0, O_q) \longrightarrow C^*(M)$$

$$\phi_{\mathcal{H}} : H^*(\mathcal{X}_0, O_q) \longrightarrow H^*(B\overline{\Gamma_q})$$

Canonical forms

$\mathcal{F}$ foliation of codim q

$f_i : U_i \longrightarrow \mathbb{R}^q$ submersion

$\gamma_{ij} : U_i \cap U_j \longrightarrow \Gamma_q$ = pseudo group of germs

$E' \subset TM$ $\qquad E'|_{U_i} = \ker df_i'$ $\qquad E'$ is integrable.

Let $Q = TM/_E$, Principle $Gl_q(\mathbb{R})$ - bdle $\pi: P(Q) \longrightarrow M$.

on $P(Q)$ we have canonical 1-forms $\omega_1, -, \omega_q$ corresponding to canonical trivialization. $\pi^*(Q^*) = \text{span}(\omega_1, -, \omega_q)$.

they generate an ideal in $\Omega^1(P(Q))$, Hence there're forms $\phi_{ij} \in \Omega^1(P(Q))$ such that

$$d\omega_i = \sum_j \underset{\uparrow}{\phi_{ij}} \wedge \omega_j$$

these are not unique.

$P_k(Q)$ = bundle on M whose fibers over x is the space of k-jets of all submersions $f_u : U \longrightarrow \mathbb{R}^q$.

$$\text{———} \xrightarrow{\pi_2} P_2(Q) \xrightarrow{\pi_1} P_1(Q) \xrightarrow{\pi} M$$

A point in $P^1(Q)$:

$$J^1(f) = (x : \partial f/_{\partial x_1} , \quad , \partial f/_{\partial x_q}) \qquad x \in M \qquad f : U \longrightarrow \mathbb{R}^q$$

X tangent to $P(Q)$ at $J^1(f)$,

one parameter family of jets $J^1(f_t)$ $\quad f_t : U \longrightarrow \mathbb{R}^q$. Thus $\left.\frac{\partial f_t}{\partial t}\right|_{t=0} = \frac{\partial f}{\partial x_j} \cdot \frac{\partial x_j}{\partial t}$

$\omega_j(X) = \partial x_j/_{\partial t}$ = j th component of $\pi_*(X)$ in the frame $\left(\frac{\partial f}{\partial x_1}, -, \frac{\partial f}{\partial x_q}\right)$

Next, suppose Y is tangent to $P_2(Q)$ at $J^2(f)$. It's again given by the two jets of a 1-parameter family, $f_t : U \to \mathbb{R}^q$ and we set

$$\theta_{ij}(Y) = \frac{\partial}{\partial x_j}\left(\frac{\partial x_L}{\partial t}\right)$$

by calculus:

$$d\omega_i = \sum \theta_{ij} \wedge \omega_j \qquad \text{(suppressing } \pi_2^*$$

Similarly on $P_3(Q)$ we can define q^3-forms: $\theta_{ijk} \in \Omega^1(P_3(Q))$

$$d\theta_{ij} = \sum \theta_{ijk} \wedge \omega_k + \sum \theta_{ik} \wedge \theta_{kj}$$

Thus if we pass to the space $P_\infty(Q) = \varprojlim P_k(Q)$

we'll have canonical one-forms satisfying local identities

~~formy~~ We're looking for universal algebra with these identities:

$\mathbb{R}^q[[x_1, \dots, x_q]]$ formal power series

L_q denote the free module on q generators, $\partial/\partial x_1, \dots, \partial/\partial x_q$

over this ring.

Lie-structure $\left[\sum f_i \partial/\partial x_i, \sum g_j \partial/\partial x_j\right] = \sum_j \left[\sum_i f_i \frac{\partial g_j}{\partial x_i} - g_i \frac{\partial f_j}{\partial x_i}\right] \frac{\partial}{\partial x_j}$

there is top on L_q called $\mathfrak{m}$-adic top

maximal ideal of $\mathbb{R}^q[[x_1, \dots, x_q]]$

slogan: L_q = "Lie algebra of Γ_q".

filtering: $-1 \leq k \leq \infty$ $\quad L_q^{-1} = \{\partial/\partial x_i\}$, $L_q^{0} = \{x_i \partial/\partial x_j\}$, $L_q^{1} = \{x_i x_j \partial/\partial x_k\}$

$[L_q^k, L_q^l] \subset L_q^{k+l}$. L_q^{-1} is abelian sub algebra of dim q.

L_q^0 is subalg gl_q. $GL_q \subset T_q$.

L_q^0 acts on $L_q^k \rightsquigarrow GL_q$ acts on $S^k(\mathbb{R}^q)^* \otimes \mathbb{R}^q$.

$L_q^* = \{ f : \mathcal{L}_q \longrightarrow \mathbb{R} \mid f$ is linear, continuous in M-adic top$\}$.

ω_i spanning the dual L_q^{-1}

θ_{ij} 〃 〃 〃 L_q^0

$$d\omega_i = \sum \theta_{ij} \wedge \omega_j \qquad d\theta_{ij} = \sum \theta_{ijk} \wedge \omega_k + \sum \theta_{ik} \wedge \theta_{kj} \cdots$$

<u>Def</u>, $\Omega^p(P_\infty(\Phi)) = \lim_n \Omega^p(P_n(\Phi))$

$$\Phi, \quad \Lambda^*(L_q^*) \longrightarrow \Omega^*(P_\infty(\Phi))$$

Universal $\omega_i \longrightarrow \omega_i$.

〃 $\theta_{ij} \longrightarrow \theta_{ij}$.

induces $H^*(\mathcal{L}_q^*) \longrightarrow H^*(P_\infty(\Phi)) \cong H(P_1(\Phi))$

$(P_{k+1}(\Phi) \longrightarrow P_k(\Phi))$

$\nearrow$ fibers are contractible

$$H^*(\mathcal{L}_q) \longrightarrow H^*(P_1(\Phi)) \xrightarrow{\text{section}} H^*(M)$$

if Φ is trivial

$P_1(\Phi) \longrightarrow M$ has section if not $P_1(Q)/O_q \sim M$

$$\begin{tikzcd}
H^*(\mathcal{L}_q) \arrow[r] & H^*(P_1(Q)) \\
H^*(L_q, O_q) \arrow[u] \arrow[r] & H^*(M) \arrow[u]
\end{tikzcd}$$

$$H^*(\mathcal{L}_q, O_q) \cong H^*(WO_q)$$

Relation to GV-classes ; $H^*(\mathcal{L}_1) = ?$

$$Y_k = x^{k+1}\frac{\partial}{\partial x} \qquad -1 \leq k \qquad , \qquad [Y_i, Y_j] = (j-i)\, Y_{i+j}$$

so if $\underset{\substack{\uparrow \\ Y_{-1}}}{\alpha_{-1}}, \underset{\substack{\uparrow \\ Y_0}}{\alpha_0}, \alpha_1, \ldots$ basis for L_1^*

$$d\alpha_k = \sum_{i<k-i} (k-2i)\, \alpha_i \wedge \alpha_{k-i} \quad \text{in } \Lambda^2(L_q^*)$$

d (higher exterior) is complicated.

Trick : Study the adjoint action of Y_0 on $\Lambda^*(L_1^*)$

$[Y_0, Y_j] = j\, Y_j$. Shows that α_j is an eigenvector for $ad(Y_0)$ of eigenvalue j.

Hence, $\alpha_{j_1} \wedge \cdots \wedge \alpha_{j_p}$ is an eigenvector for $ad(Y_0)$ of eigenvalue

$$\sum_{i=1}^{n} j_i \qquad \Lambda^*(L_1^*) = \sum_{r=-1}^{\infty} C_r^* \longrightarrow$$ is eigenspaces of eigenvalues r.

C_r^* is acyclic if $r \neq 0$. if $\beta \in C_r^*$ is a cocycle and $r \neq 0$.

$$r\beta = \underset{\substack{\uparrow \\ \text{Lie} \\ \text{derivative}}}{ad(Y_0)}\beta = d\iota_{Y_0}(\beta)$$, $\iota_{Y_0}\beta \in C_r^*$, Thus $H^*(\Lambda^*(L_1^*)) \cong H^*(C_0^*)$.

Now a monomial $\alpha_{j_1} \wedge \cdots \wedge \alpha_{j_p}$ can have eigenvalue zero only if $p \leq 3$:

C_0^* is zero in degrees $p \geq 4$. and is generated by

1, $C_0^1 = \{\alpha_0\}$, $C_0^2 = \{\alpha_{-1} \wedge \alpha_1\}$ and $C_0^3 = \{\alpha_{-1} \wedge \alpha_0 \wedge \alpha_1\}$

$d\alpha_0 = 2\,\alpha_{-1} \wedge \alpha_1$ $\qquad \alpha_0$ is not cycle

$\alpha_{-1} \wedge \alpha_1$ is a coboundary. Thus $H^*(L_1) = H^*(C_0^*) = \Lambda(y)$

, deg $y = 3$. y is represented by form $\alpha_{-1} \wedge \alpha_0 \wedge \alpha_1$.

GV in Codim 1: $E \subset TM$ be codim 1. $Q = TM/E$

on $P_1(Q)$ we have canonical form ω, on $P_2(Q)$ a canonical form θ such that $d\omega = \theta \wedge \omega$ & on $P_3(Q)$ we have θ_1

$d\theta = \theta_1 \wedge \omega$ and so on.

$$\phi : \mathcal{L}_1^* \longrightarrow \Omega^*(P_\infty(Q))$$

$$\phi(\alpha_{-1}) = \omega \,,\quad \phi(\alpha_0) = \theta \,,\quad \phi(\alpha_k) = \theta_k \quad \text{for } k \geq 1$$

$$\phi^*(\alpha_{-1} \wedge \alpha_0 \wedge \alpha_1) = \{\omega \wedge \theta \wedge \theta_1\} = \{\theta \wedge d\theta\}$$