Perfectness of $\mathrm{Diff}_c^\infty(M)$

C^∞ diffeos, cptly supported isotopic to id via cptly supp. isotopy

References: Banyaga's book, Thurston's paper(?), Haller-Rybicki-Teichmann (2012), my exposition.

Some history - Algebraic structure of Diff

- 1960's Smale conjectures that $\mathrm{Diff}_c^r(M)$ is simple. (Known for Homeo)
- 1970's Epstein: commutator subgroup is simple

Q: Is Diff = commutator subgroup

$$H_1(\underset{\text{as discrete group}}{\mathrm{Diff}};\mathbb{Z}) = \mathrm{Diff}/\text{commutator} \overset{?}{=} 0$$

- Herman: True for $\mathrm{Diff}_c^\infty(\mathbb{T}^n)$ → version of Nash-Moser implicit fn. thm for Frechet manifolds (Sergeraert)

Precise statement of Herman's theorem, $n=1$ case:

Thm: $\exists$ a neighborhood $\mathcal{U}$ of id in $\mathrm{Diff}_0^\infty(S^1)$ and rotation R_θ s.t. any $g = \mathcal{U}$ can be written $g = R_\lambda \circ [g', R_\theta]$

("implicit/inverse function thm": $S^1 \times \mathrm{Diff}_0^\infty(S^1) \longrightarrow \mathrm{Diff}_0^\infty(S^1)$
$(\lambda, g') \longmapsto R_\lambda \circ [g', R_\theta]$
has a "smooth" ~~inverse~~ local inverse in nbd of id.)
Diff^∞ is a frechet space

Facts: • any rotation (i.e. R_λ) can be written as commutator (we'll do this later)
• nbd of id generates Diff_0

Conclude: $\mathrm{Diff}_0^\infty(S^1)$ is perfect.

- Thurston: True for general M.

Thm: Let $B \hookrightarrow M^n$ be inclusion of open ball. Then $\mathrm{Diff}_c^r(B) \to \mathrm{Diff}_c^r(M)$ induces isomorphism $H_1(\mathrm{Diff}_c^r(M);\mathbb{Z}) \cong H_1(\mathrm{Diff}_c^r(B);\mathbb{Z})$

Herman + Thurston $\Rightarrow$ $H_1(\mathrm{Diff}_c^\infty(M^n)) = H_1(\mathrm{Diff}_c^\infty(B)) = H_1(\mathrm{Diff}_c^\infty(\mathbb{T}^n)) = 0$.

Thurstons proof uses relationship with haefliger structures/classifying spaces for foliations. We'll see a related theorem (of Mather, proof due to Segal) next time.

<u>Today:</u> "hands-on approach" from Haller - Rybicki - Teichmann

I. Reduction to $M = \mathbb{R}^n$ via fragmentation (common technique)

<u>Fragmentation lemma</u>: let $\{U_\alpha\}$ be an open cover of M

Any $g \in \mathrm{Diff}_c^\infty(M)$ can be written as $g = g_1 g_2 \cdots g_k$ where each g_i is supported in some U_α. [Proof is not hard - see Banyaga or my paper]

<u>Upshot</u>: if $\mathrm{Diff}_c^\infty \mathbb{R}^n$ is perfect, so is $\mathrm{Diff}_c^\infty(M)$.

- Can also reduce to nbd of id in $\mathrm{Diff}_c^\infty(\mathbb{R}^n)$ since nbd of id generates

<u>Thm</u> (H-R-T): $\exists$ VF's $X_1, X_2, \ldots X_m$ on $\mathbb{R}^n$ s.t. $(n \geq 2)$

$$\mathrm{Diff}_c^\infty(\mathbb{R}^n) \times \cdots \times \mathrm{Diff}_c^\infty(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_c^\infty(\mathbb{R}^n)$$

$$(g_1, g_2, \ldots g_m) \longmapsto [g_1, \exp(X_1)] \circ \cdots \circ [g_m, \exp(X_m)]$$

has smooth local inverse at id.

- note parallel with Herman's thm (which is used (T^n version) in their proof)
- Actually, H-R-T prove that you can use just 4 vector fields ~~&~~ for any manifold M !!

<u>Today we'll prove</u>: (weaker version with easier proof)

<u>Thm</u>: $\exists$ m depending on n and nbd U of id in $\mathrm{Diff}_c^\infty \mathbb{R}^n$ $(n \geq 2)$ s.t. any $g \in U$ can be written $g = [g_1, g_2] \circ \cdots \circ [g_{2m-1}, g_{2m}]$ with g_i depending smoothly on g.

- via "induction on dimension"
- using only Herman for S^1

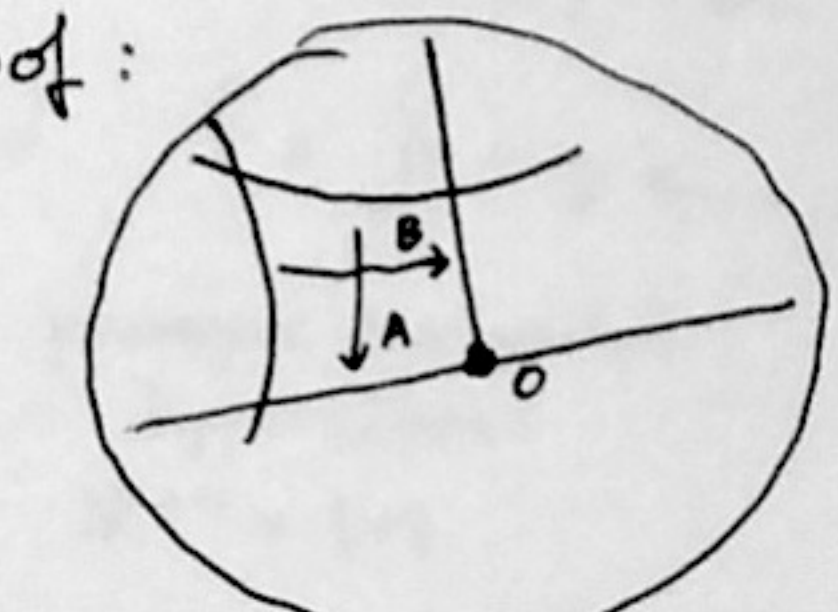
II. Work for base case

- **Prove thm for $\mathrm{Diff}_0^\infty(S^1)$:**

given Herman, we just need to write R_λ as a commutator (in a way that smoothly depends on λ, locally).

Hyperbolic geometry proof:

(can do it in $PSL_2\mathbb{R}$)

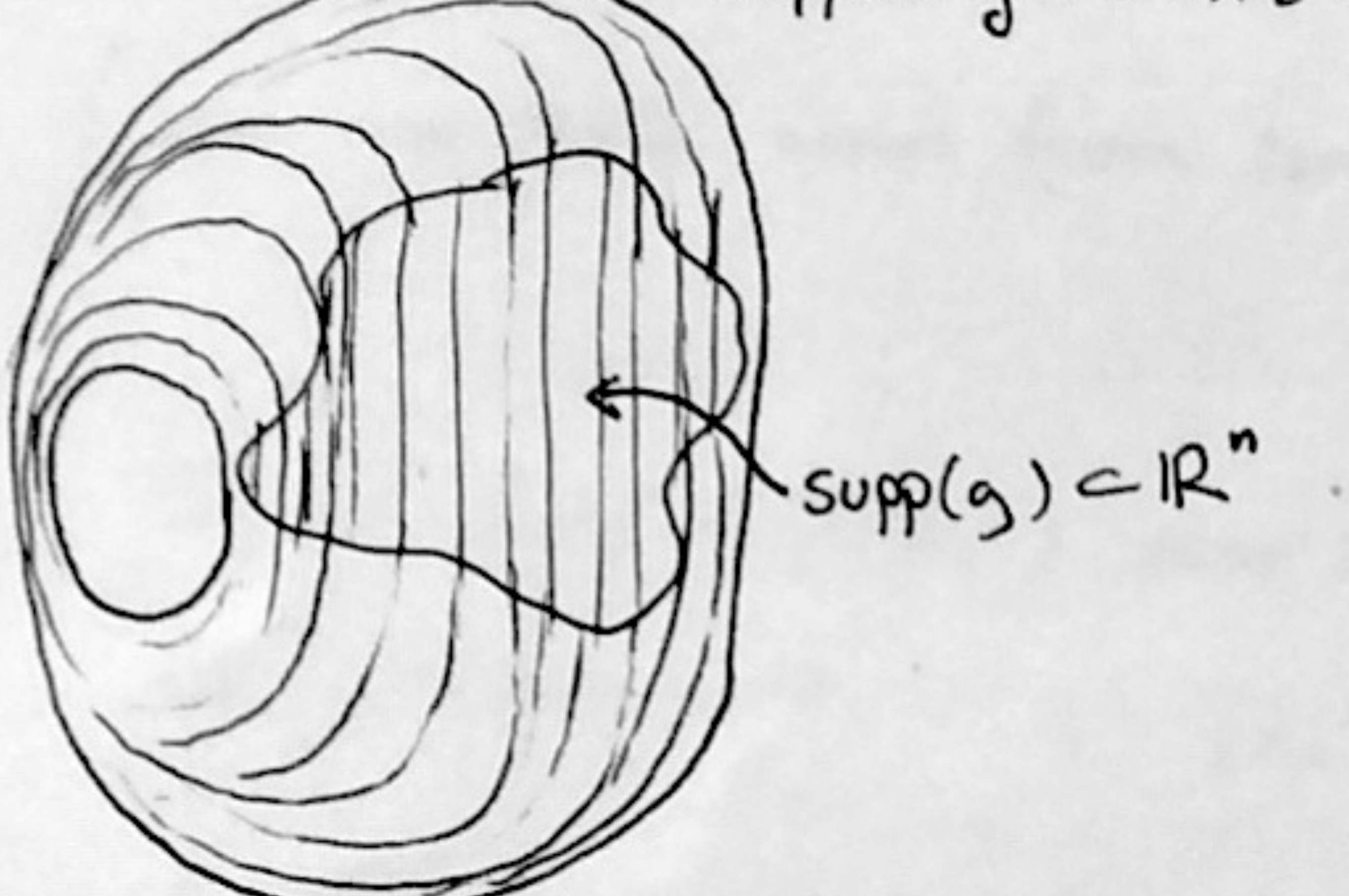


- $ABA^{-1}B^{-1}$ fixes O
- is rotation by $2\pi -$ (angle sum of ▢)

Less enlightening proof: write down some matrices.

- **Prove thm for Diffeos preserving vertical lines:**

Suppose $g \in \mathrm{Diff}_c^\infty(\mathbb{R}^n)$ preserves each vertical line & is close to id. We'll show how to write as commutator.



$S^1 \times [0,1]^{n-1}$ foliated by circles $S^1 \times \{t\}$.

1. extend vertical lines that meet $\mathrm{supp}(g)$ to a foliation by circles: i.e. embedded $S^1 \times [0,1]^{n-1}$ containing $\mathrm{supp}(g)$ s.t. g preserves each $S^1 \times \{t\}$.

Let g_t denote restriction of g to $S^1 \times \{t\}$. (Close to id. if g is).
as diffeos of S^1, these vary smoothly in t.

By Herman, $g_t = [a_t, b_t][g'_t, R_\theta]$

Since a_t, b_t, g'_t vary smoothly in t, they glue together to form smooth diffeos defined on $S^1 \times [0,1]^{n-1} \subset \mathbb{R}^n$.

Since $g|_{\partial(S^1 \times [0,1]^{n-1})} = \mathrm{id}$, a_t, b_t, g'_t are id on $\partial(S^1 \times [0,1]^{n-1})$

so extend to diffeos of $\mathbb{R}^n$ supported on cpt set $S^1 \times [0,1]^{n-1}$.

Extend R_θ to $r \in \mathrm{Diff}_c^\infty(\mathbb{R}^n)$ by isotoping to id on slightly bigger embedded $S^1 \times D$.

Conclude: $g = [a,b][g',r]$; a, b, g', r can be made to smoothly depend on g.

III. Inductive argument.

Lemma: $\exists$ nbd $\mathcal{U}$ of id in $\mathrm{Diff}_c^\infty(\mathbb{R}^n)$ s.t. any $f \in \mathcal{U}$ can be factored $f = h \circ g$

h preserves horizontal hyperplanes $\mathbb{R}^{n-1} \times \{y\}$

g preserves vertical lines.

in a natural way.

Proof of thm given lemma:

Assume inductively that cptly supported diffeos of $\mathbb{R}^{n-1}$ near id can be written as product of commutators, "smoothly"

(base case $n=2$ comes from lemma + diffeos preserving vertical lines in $\mathbb{R}^2$ (g)
+ diffeos presrvg horizontal lines in $\mathbb{R}^2$ (h))
(same thing.)

Given $f \in \mathrm{Diff}_c^\infty(\mathbb{R}^n)$ near id, use lemma to factor $f = h \circ g$.

By previous work, $g = [a,b][g',r]$ (depending smoothly on g)

By hypothesis, restriction of h to each plane $\mathbb{R}^{n-1} \times \{y\}$ (call this h_y) can be written as product of commutators in $\mathrm{Diff}_c^\infty(\mathbb{R}^{n-1})$, smoothly depending on y $\Rightarrow$ piece together to give globally defined factorization of h as product of commutators. $\square$