Math 185-1

Midterm 2

March 31, 2016

Name: ______________________________

- You will have **75 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ______________________________

1. Determine whether the following statements are true or false. No justification is required.

   (a) (2 points) If $u(z)$ is a harmonic function on a disk $\{z : |z - z_0| < \rho\}$ and $r < \rho$, then the average value of $u(z)$ on the circle $\{z : |z - z_0| = r\}$ is equal to $u(z_0)$.

   TRUE false

   (b) (2 points) The function $\cos z$ has both 2π and $2\pi i$ as periods.

   true **FALSE**

   (c) (2 points) Let $f(z)$ be an analytic function on a domain D. Then $\int_\gamma f(z)dz$ is independent of path on D.

   true **FALSE**

   (d) (2 points) If $f(z)$ and $g(z)$ are entire functions and $f(z) = g(z)$ for all z such that $|z| < 1$, then $f(z) = g(z)$ for all $z \in \mathbb{C}$.

   TRUE false

   (e) (2 points) If $f(z)$ is (complex) differentiable at every point in a domain D, then $f(z)$ has derivatives of all order at every point in D.

   TRUE false

2. (10 points) Compute

$$\oint_{|z|=2} \frac{\cos z}{z(z-\frac{\pi}{2})^2} dz.$$

Solution: Let $\epsilon > 0$ be small, and let D_ϵ be the disk of radius 2 centered at the origin with disks of radius ϵ around 0 and $\frac{\pi}{2}$ removed. Then, $\frac{\cos z}{z(z-\frac{\pi}{2})^2}$ is analytic on D_ϵ, so by Cauchy's theorem, $\int_{\partial D_\epsilon} \frac{\cos z}{z(z-\frac{\pi}{2})^2} dz = 0$.

But ∂D_ϵ consists of the circle $|z| = 2$ oriented counter-clockwise and the circles $|z| = \epsilon$ and $|z - \frac{\pi}{2}| = \epsilon$ oriented clockwise. Therefore,

$$\int_{|z|=2} \frac{\cos z}{z(z-\frac{\pi}{2})^2} dz - \int_{|z|=\epsilon} \frac{\cos z}{z(z-\frac{\pi}{2})^2} dz - \int_{|z-\frac{\pi}{2}|=\epsilon} \frac{\cos z}{z(z-\frac{\pi}{2})^2} dz = 0.$$

The integrals around the disks of radius ϵ can be evaluated using Cauchy's integral formula, as $f_1(z) = \frac{\cos z}{(z-\frac{\pi}{2})^2}$ is analytic on the disk $|z| \leq \epsilon$ and $f_2(z) = \frac{\cos z}{z}$ is analytic on the disk $|z - \frac{\pi}{2}| < \epsilon$.

We compute,

$$\int_{|z|=\epsilon} \frac{\cos z}{z(z-\frac{\pi}{2})^2} dz = 2\pi i f_1(0) = 2\pi i \left(-\frac{2}{\pi}\right)^2 = \frac{8i}{\pi},$$

and

$$\int_{|z-\frac{\pi}{2}|=\epsilon} \frac{\cos z}{z(z-\frac{\pi}{2})^2} dz = 2\pi i f_2'(\frac{\pi}{2}) = 2\pi i \left(\frac{-\pi/2}{(\pi/2)^2}\right) = -4i.$$

Thus,

$$\int_{|z|=2} \frac{\cos z}{z(z-\frac{\pi}{2})^2} dz = \int_{|z|=\epsilon} \frac{\cos z}{z(z-\frac{\pi}{2})^2} dz + \int_{|z-\frac{\pi}{2}|=\epsilon} \frac{\cos z}{z(z-\frac{\pi}{2})^2} dz = \frac{8}{\pi} i - 4i.$$

3. (10 points) Let γ be a piecewise smooth curve of finite length in $\mathbb{C}$, and suppose that $\{f_j\}_{j=0}^{\infty}$ is a sequence of continuous complex-valued functions on γ that converges uniformly to f on γ. Use the ML-estimate to show that $\int_\gamma f_j(z)dz$ converges to $\int_\gamma f(z)dz$.

Solution: Suppose γ has length L and let $\epsilon > 0$.

By uniform convergence, there exists an N such that $n > N$ implies that $|f_n(z) - f(z)| < \epsilon/L$ for all $z \in \gamma$. Using the ML-estimate on $f_n(z) - f(z)$, we have that

$$\begin{aligned} |\int_\gamma f_n(z)dz - \int_\gamma f(z)dz| &= |\int_\gamma [f_n(z) - f(z)]dz| \\ &\leq \frac{\epsilon}{L}L = \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz$.

4. (10 points) Compute the terms up to order 5 (i.e. up to z^5) of the power series centered at $z = 0$ for the function $f(z) = \frac{\cos z - 1}{1+z}$ (you do not have to simplify your constants, i.e. you may leave them in the form $c_1 + c_2 + c_3 + \dots$ without computing the value of $c_1 + c_2 + c_3 + \dots$). Determine the radius of convergence of the power series and the order of the zero of $f(z)$ at 0.

Solution: We have that $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$. This can be derived using the derivatives of $\cos z$, which cycle $-\sin z, -\cos z, \sin z, \cos z, -\sin z, \dots$.

Then, $\cos z - 1 = -\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$.

We can also find a power series for $\frac{1}{1+z}$ for $|z| < 1$ as a geometric series with common ratio $-z$, i.e. $\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - z^5 + \dots$.

Then, $f(z)$ is the product of the power series for $\cos z - 1$ and $\frac{1}{1+z}$, so

$$\begin{aligned} f(z) &= (\cos z - 1)\left(\frac{1}{1+z}\right) \\ &= \left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right)\left(1 - z + z^2 - z^3 + z^4 - z^5 + \dots\right) \\ &= -\frac{z^2}{2} + \frac{z^3}{2} + \left(-\frac{1}{2} + \frac{1}{24}\right)z^4 + \left(\frac{1}{2} - \frac{1}{24}\right)z^5 + \dots \\ &= -\frac{1}{2}z^2 + \frac{1}{2}z^3 - \frac{11}{24}z^4 + \frac{11}{24}z^5 + \dots, \end{aligned}$$

where we multiply power series using uniform convergence of the individual power series.

Since $f(z)$ cannot be made analytic at $z = -1$, the radius of convergence of the power series is the distance from 0 to -1, which is 1. The order of the zero at $z = 0$ is 2, since $a_0 = a_1 = 0$, but $a_2 = -1/2 \neq 0$.

5. (a) (5 points) Suppose that $f(\infty) = c$ is finite and $f(z)$ is analytic at ∞. Show that there exists a $\rho > 0$ such that $|f(z)| < |c| + 1$ for all $|z| > \rho$.

> **Solution:** Since $f(z)$ is analytic at ∞, then $g(w) = f(1/w)$ is analytic at $w = 0$, where $w = 1/z$. In particular, $g(w)$ is continuous at 0, so letting $\epsilon = 1$, by continuity, there exists $\delta > 0$ such that $|w - 0| < \delta$ implies that $|g(w) - g(0)| < 1$. In other words, $|g(w)| - |c| < 1$ by the reverse triangle inequality, or $|g(w)| < |c| + 1$.
>
> Then, taking $\rho = \frac{1}{\delta}$, we have that if $|z| > \rho$, then $|w| = \frac{1}{|z|} > \frac{1}{\rho} = \delta$, so $|f(z)| = |g(w)| < |c| + 1$, as desired.

(b) (5 points) Show that if $f(z)$ is an entire function such that $f(\infty)$ is finite and $f(z)$ is analytic at ∞, then $f(z)$ is constant on $\mathbb{C} \cup \{\infty\}$.

> **Solution:** By the previous part, letting $c = f(\infty)$, there exists an $\rho > 0$ such that $|f(z)| < |c| + 1$ for all $|z| > \rho$. The disk $\{z : |z| \leq 2\rho\}$ is compact, and $f(z)$ is analytic, hence continuous on the disk, so the image of $f(z)$ on the disk is bounded, i.e. there exists M such that $|f(z)| \leq M$ for all $|z| \leq 2\rho$. Then, $\max\{M, |c| + 1\}$ is a bound for $|f(z)|$ for all $z \in \mathbb{C}$, and $f(z)$ is entire, so by Liouville's theorem, $f(z) = a$ is constant on $\mathbb{C}$. As $g(w) = f(1/w)$ is analytic at $w = 0$, it is also continuous at $w = 0$.
>
> Since $g(w) = a$ for all $w \neq 0$ (as $z = 1/w$ is in $\mathbb{C}$ and $f(z) = a$ for all z), so it must be that $a = c$, so $f(z) = c$ for all $z \in \mathbb{C} \cup \{\infty\}$.

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Question:	1	2	3	4	5	Total
Points:	10	10	10	10	10	50
Score:						