Math 185-1

Midterm 1

February 18, 2016

Name: ____

- You will have **70 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ____

1. Determine whether the following statements are true or false. No justification is required.

   (a) (2 points) The function $\operatorname{Arg} z$ is continuous on $\mathbb{C} \setminus \{0\}$.

   true **FALSE**

   (b) (2 points) If $f : \mathbb{C} \to \mathbb{C}$ and $\lim_{\epsilon \to 0} f(z_0 + \epsilon) = \lim_{\epsilon \to 0} f(z_0 + i\epsilon)$, then $\lim_{z \to z_0} f(z)$ exists.

   true **FALSE**

   (c) (2 points) If $Pdx + Qdy$ is an exact differential on a domain D, then $Pdx + Qdy$ is closed on D.

   TRUE false

   (d) (2 points) If a domain D is star-convex, then D is convex.

   true **FALSE**

   (e) (2 points) If f is analytic on a domain D, $z_0 \in D$, and $f'(z_0) \neq 0$, then f is conformal at z_0.

   TRUE false

2. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence expressed in the form $z_n = r_n e^{i\theta_n}$.

   (a) (8 points) Show that if $\lim_{n\to\infty} r_n = r$ and $\lim_{n\to\infty} \theta_n = \theta$ exist, then $\lim_{n\to\infty} z_n = re^{i\theta}$.

   Solution: Let $z_n = r_n e^{i\theta_n}$. Then, $z_n = r_n \cos\theta_n + ir_n \sin\theta_n$. Since cosine and sine are continuous functions, as $\theta_n \to \theta$, we have that $\cos\theta_n \to \cos\theta$ and $\sin\theta_n \to \sin\theta$.

   Therefore, as $n \to \infty$, $r_n \cos\theta_n \to r\theta$ and $r_n \sin\theta_n \to r\theta$ since products of convergent series converge to the product of the limits.

   Thus, the real and imaginary parts of the sequence $\{z_n\}$ converges, so that $z_n \to r\cos\theta + ir\sin\theta = re^{i\theta}$.

   (b) (2 points) If $\lim_{n\to\infty} z_n$ exists, must $\lim_{n\to\infty} r_n$ and $\lim_{n\to\infty} \theta_n$ exist? Briefly explain why (a formal proof is not required).

   Solution: Not necessarily. For example, take $z_n = 1$ with $r_n = 1$ and $\theta_n = 2\pi n$ so that $z_n = r_n e^{i\theta_n}$. Since z_n is a constant sequence, $z_n \to 1$, but θ_n is a divergent series.

3. Suppose that $u(x, y)$ is a real-valued harmonic function on a domain D with continuous partial derivatives.

   (a) (5 points) Show that
$$-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$
is a closed differential.

Solution: In order for the differential to be closed, we must have that
$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \triangle u,$$
is equal to 0. Since u is harmonic, we have that $\triangle u = 0$, so that the differential is closed.

   (b) (5 points) Suppose that $v(x, y)$ is a real-valued harmonic conjugate to $u(x, y)$ on D with continuous partial derivatives. Show that
$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

Solution: First, we know that
$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy.$$
Since v is a harmonic conjugate of u, we have that u, v satisfy the Cauchy-Riemann equations, so that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Substituting into the formula for dv gives
$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

4. (a) (4 points) Let $f(z) = z^2$. Use the definition of the derivative to show that f is analytic on $\mathbb{C}$, and find the derivative. (You may instead use the Cauchy-Riemann equations for partial credit.)

Solution: By definition,

$$\begin{aligned} f'(z) &= \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \\ &= \lim_{\delta z \to 0} \frac{(z+\delta z)^2 - z^2}{\delta z} \\ &= \lim_{\delta z \to 0} \frac{z^2 + 2z\delta z + \delta z^2 - z^2}{\delta z} \\ &= \lim_{\delta z \to 0} 2z + \delta z \\ &= 2z. \end{aligned}$$

In particular, $f'(z) = 2z$ exists for all $z \in \mathbb{C}$, and $2z$ is continuous, so that f is analytic on $\mathbb{C}$.

(b) (3 points) Define a single-valued (i.e. branch) inverse function $\sqrt{z}$. Be clear to explain your domain, the value of the inverse function on your domain, and the points of discontinuity (i.e. branch cut).

Solution: Let $w = z^2$. Then, if $w = re^{i\theta}$, we can see that $z = r^{1/2}e^{i\theta/2}$. So to define an inverse function f^{-1}, we set

$$f^{-1}(z) = |z|^{1/2} e^{i \operatorname{Arg} z/2},$$

if $z \neq 0$, and $f^{-1}(0) = 0$ as the inverse. This is define for all $\mathbb{C}$, and discontinuous on $(-\infty, 0)$ because of the discontinuity of $\operatorname{Arg} z$. (Note that other choices are possible, and they will essentially just depend on the branch of the argument function that you choose.)

(c) (3 points) Find the derivative of your inverse function $\sqrt{z}$.

Solution: Since $f'(z) = 2z \neq 0$ for $z \neq 0$, by the inverse function theorem, letting $w = z^2$,

$$(f^{-1})'(w) = \frac{1}{f'(z)} = \frac{1}{2z} = \frac{1}{2\sqrt{w}},$$

where $\sqrt{w}$ is defined using the inverse from part (b). Note that this is really defined only on $\mathbb{C} \setminus (-\infty, 0]$ because of the discontinuity of the $\sqrt{w}$ function, although this formula still holds on all of $\mathbb{C} \setminus \{0\}$ by taking different branches of the square root function.

5. Suppose f is a fractional linear transformation that maps 0 to 1, the x-axis to the circle $\{z : |z-(1+i)| = 1\}$, and the unit circle $\{z : |z| = 1\}$ to the horizontal line $y = 1$.

   (a) (5 points) Determine the image of the y-axis under the transformation f.

   Solution: The image of the y-axis passes through 1 since $f(0) = 1$, and it is perpendicular to both $y = 1$ (as the y-axis is perpendicular to its preimage $|z| = 1$) and to the circle $|z-(1+i)| = 1$ (as the y-axis is perpendicular to its preimage, the x-axis). This is true because fractional linear transformations are conformal maps, so must present angles.

   Since the image of an extended circle is an extended circle, the only option is for the image of the y-axis to be the vertical line $x = 1$.

   (b) (5 points) Determine the shape of the image of the vertical line $x = 2$ under f, and describe the number of intersections and position relative to the circle $\{z : |z-(1+i)| = 1\}$ and the horizontal line $y = 1$.

   Solution: Since the circle $|z| = 1$ maps the line $y = 1$, and ∞ lies on the extended circle given by the line $y = 1$, this means that $f^{-1}(\infty)$ is somewhere on the circle $|z| = 1$. In particular, since the line $x = 2$ does not intersect the circle $|z| = 1$, the preimage of ∞ does not lie on line $x = 2$, so the image of the line $x = 2$ must be a circle (it is an extended circle that does not pass through ∞). It does not intersect the horizontal line $y = 1$ because its preimage, the circle $|z| = 1$, does not intersect the line $x = 2$. Moreover, we can see that the image of $x = 2$ must lie above the line $y = 2$. This is because the interior of the disk $|z| < 1$, which contains 0, must map to below the line $y = 2$, which contains $f(0) = 1$.

   The image of the line $x = 2$ under f also intersects the circle $|z-(1+i)| = 1$ orthogonally twice, since its preimage (the x-axis) intersects $x = 2$ at 2 and ∞, both orthogonally.

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Question:	1	2	3	4	5	Total
Points:	10	10	10	10	10	50
Score:						