Math 185 Lecture 4
Final Exam
May 14, 2014

Name: ____________________

- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- You will have **150 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please do not detach the formula sheet from the exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ____________________

Stereographic projection:

$$x = X/(1-Z) \qquad\qquad X = 2x/(|z|^2+1)$$
$$y = Y/(1-Z) \qquad\qquad Y = 2y/(|z|^2+1)$$
$$Z = (|z|^2-1)/(|z|^2+1).$$

Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Harmonic conjugate:

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s,y_0)ds + C$$
$$v(B) = \int_A^B -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

Fractional linear transformation:

$$w = f(z) = \frac{z-z_0}{z-z_2}\frac{z_1-z_2}{z_1-z_0}.$$

Mean value property:

$$u(z_0) = \int_0^{2\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi}.$$

Cauchy integral formula:

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}dw.$$

Power series and Laurent series:

$$a_k = \frac{1}{2\pi i}\oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}d\zeta.$$

Residue theorem:

$$\int_{\partial D} f(z)dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}[f(z), z_j].$$

Argument principle:

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi}\int_{\partial D} d\arg(f(z)) = N_0 - N_\infty.$$

Inverse function theorem:

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{|\zeta - z_0| = \rho} \frac{\zeta f'(\zeta)}{f(\zeta) - w}d\zeta, \qquad\qquad |w - w_0| < \delta.$$

1. For each of the following functions, determine whether the given point is a removable singularity, a pole, an essential singularity, or not a singularity. If it is a pole, give the order of the pole.

   (a) (3 points) $z^{-1} \cos \frac{1}{z}$ at $z = 0$

   Solution: The power series for $\cos z$ centered at 0 converges for all z and is given by
   $$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots.$$
   Therefore, for $z \neq 0$,
   $$\cos \frac{1}{z} = 1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} + \dots.$$
   Then,
   $$z^{-1} \cos z = \frac{1}{z} - \frac{1}{2!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^5} + \dots$$
   so that $z = 0$ is an essential singularity.

   (b) (3 points) $\frac{1-\cos z}{z^3(z-\pi)}$ at $z = 1$

   Solution: The function is analytic at $z = 1$, so it is not a singularity.

(c) (3 points) $\frac{(z-3)\sin(\pi z)}{z(z-1)^3}$ at $z = 1$

Solution: We have that $\sin \pi = 0$ but the first derivative $\sin(\pi z)' = \pi \cos(\pi z)$ is non-zero at $z = 1$. Hence, $z = 1$ is a zero or order 1 for $(z-3)\sin(\pi z)$. Since $z = 1$ is a zero of order 3 for the denominator $z(z-1)^3$, then $\frac{(z-3)\sin(\pi z)}{z(z-1)^3}$ has a pole of order 2 at $z = 1$.

(d) (3 points) $\frac{z(z-1)^3}{(z-3)\sin(\pi z)}$ at $z = 1$

Solution: This is the reciprocal of part (c), which has a pole of order 2 at $z = 1$. Hence, $\frac{z(z-1)^3}{(z-3)\sin(\pi z)}$ has a removable singularity at $z = 1$, and in fact has a zero of order 2.

2. Determine whether the following statements are true or false. No justification is required.

   (a) (2 points) If $f(z)$ has an essential singularity at z_0, then $\lim_{z \to z_0} |f(z)| = \infty$.

   true **FALSE**

   (b) (2 points) If $P dx + Q dy$ is closed, then $\int P dx + Q dy$ is path independent.

   true **FALSE**

   (c) (2 points) If $f(z)$ is analytic at z_0, then $\mathrm{Res}[f(z), z_0] = 0$.

   TRUE false

   (d) (2 points) The gamma function $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ defined for $\mathrm{Re}\, z > 0$, satisfies $\Gamma(n+1) = n!$ for all positive integers n.

   TRUE false

   (e) (2 points) Every harmonic function on a domain D has a harmonic conjugate on D.

   true **FALSE**

   (f) (2 points) The number of prime numbers less than x is equal to $\log x$.

   true **FALSE**

   (g) (2 points) The power series for $\mathrm{Log}\, z$ centered at $z = -5 + i$ has radius of convergence equal to 1.

   true **FALSE**

   (h) (2 points) If $f(z)$ is a non-constant function that is analytic on D, then $f(U)$ is open for every open subset $U \subseteq D$.

   TRUE false

   (i) (2 points) The function $p(z) = z^5 + 3z^4 - 11z^3 + 4z + 2$ has 5 roots inside the unit disk $|z| < 1$.

   true **FALSE**

3. (15 points) Determine the subset of points in $\mathbb{C}$ for which $f(z) = 2z + \bar{z}^2$ is differentiable.

Solution: Let $z = x + iy$. Then

$$\begin{aligned} f(z) = f(x+iy) &= 2(x+iy) + (x-iy)^2 = 2x + 2iy + x^2 - y^2 - 2xyi \\ &= (2x + x^2 - y^2) + i(2y - 2xy). \end{aligned}$$

The real part of $f(z)$ is given by $u(x,y) = 2x + x^2 - y^2$, while the imaginary part of $f(z)$ is given by $v(x,y) = 2y - 2xy$. The function $f(z)$ is differentiable at z if and only if u and v satisfy the Cauchy-Riemann equations at z.

We see that

$$\frac{\partial u}{\partial x} = 2 + 2x \qquad \frac{\partial v}{\partial x} = -2y$$
$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2 - 2x.$$

Therefore,

$$\begin{aligned} &\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ &\Leftrightarrow 2 + 2x = 2 - 2x \\ &\Leftrightarrow x = 0, \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \\ &\Leftrightarrow -2y = -(-2y) \\ &\Leftrightarrow y = 0. \end{aligned}$$

The Cauchy-Riemann equations are satisfied if and only if $z = 0$, so $f(z)$ is differentiable if and only if $z = 0$.

4. (a) (5 points) Let Γ_R be the semicircle in the upper half-plane of radius R centered at the origin. Show that

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{iz}}{z(z^2+1)}dz = 0.$$

Solution: On Γ_R, we have that $|e^{iz}| = e^{-i\,\mathrm{Im}\,z} \le e^0 = 1$, and $|z(z^2+1)| = |z||z^2+1| \ge R(R^2-1)$. Hence,

$$\left|\frac{e^{iz}}{z(z^2+1)}\right| \le \frac{1}{R(R^2-1)} = M.$$

The length of Γ_R is $L = \pi R$. By the ML-estimate, then

$$\left|\int_{\Gamma_R}\frac{e^{iz}}{z(z^2+1)}dz\right| \le \frac{\pi R}{R(R^2-1)} = \frac{\pi}{R^2-1}.$$

As $R \to \infty$, this goes to 0, so

$$\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{iz}}{z(z^2+1)}dz = 0.$$

(b) (5 points) Let γ_ϵ be the semicircle in the upper half-plane of radius ϵ centered at the origin. Find

$$\lim_{\epsilon\to 0}\int_{\gamma_\epsilon}\frac{e^{iz}}{z(z^2+1)}dz.$$

Solution: At $z = 0$, $\frac{e^{iz}}{z(z^2+1)}$ has at worst a pole of order 1. We can compute the residue as

$$\mathrm{Res}[\frac{e^{iz}}{z(z^2+1)}, 0] = \lim_{z\to 0} z\frac{e^{iz}}{z(z^2+1)} = e^{i0}0^2+1 = 1.$$

The curve γ_ϵ travels a circular arc of angle π, so by the fractional residue theorem,

$$\lim_{\epsilon\to 0}\int_{\gamma_\epsilon}\frac{e^{iz}}{z(z^2+1)}dz = \pi i(1) = \pi i.$$

(c) (5 points) Compute

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)} dx,$$

by using the residue theorem.

Solution: Let $f(z) = \frac{e^{iz}}{z(z^2+1)}$. Then $f(z)$ has simple poles at $z = 0, \pm i$. Let $D_{R,\epsilon}$ be the half-disk of radius R in the upper half-plane, indented at 0. Only i lies inside $D_{R,\epsilon}$, so by the residue theorem,

$$\begin{aligned}\int_{\partial D_{R,\epsilon}} f(z)dz &= 2\pi i \operatorname{Res}[f(z), i] \\ &= 2\pi i \lim_{z\to i}(z-i)\frac{e^{iz}}{z(z^2+1)} \\ &= 2\pi i \lim_{z\to i} \frac{e^{iz}}{z(z+i)} \\ &= 2\pi i \frac{e^{-1}}{i(2i)} = -\frac{\pi i}{e}.\end{aligned}$$

We also have that

$$\int_{\partial D_{R,\epsilon}} f(z)dz = \int_{\Gamma_R} f(z)dz - \int_{\gamma_\epsilon} f(z)dz + \int_{\epsilon}^{R} f(z)dz + \int_{-R}^{-\epsilon} f(z)dz.$$

If we take the limit as $R \to \infty$ and $\epsilon \to 0$, applying parts (a) and (b) allows us to conclude

$$\begin{aligned}-\frac{\pi i}{e} &= 0 - \pi i + \int_0^{\infty} \frac{e^{ix}}{x(x^2+1)}dx + \int_{-\infty}^{0} \frac{e^{ix}}{x(x^2+1)}dx \\ \Rightarrow \pi i(1-e^{-1}) &= \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}dx.\end{aligned}$$

Noting that $e^{ix} = \cos x + i \sin x$, if we take the imaginary part of the above equation, we find that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}dx = \pi(1-e^{-1}).$$

5. (15 points) Find the Laurent series centered at 0 for the function

$$f(z) = \frac{1}{(z+3)(z+1)},$$

that converges at $z = 2$. Find the annulus of convergence for the Laurent series.

Solution: We first find the partial fraction decomposition

$$\frac{1}{(z+3)(z+1)} = \frac{A}{z+1} + \frac{B}{z+3}.$$

It is easy to check that $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Hence, $f(z) = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$. Using the geometric series formula, we have that for $|z| > 1$,

$$\begin{aligned}\frac{1}{z+1} &= \frac{1}{z}\frac{1}{1+1/z} \\ &= \frac{1}{z}\sum_{n=0}^{\infty}(-1/z)^n \\ &= \frac{1}{z}\sum_{n=-\infty}^{0}(-z)^n \\ &= \sum_{n=-\infty}^{-1}(-1)^{n+1}z^n.\end{aligned}$$

Also, for $|z| < 3$,

$$\begin{aligned}\frac{1}{z+3} &= \frac{1}{3}\frac{1}{1+z/3} \\ &= \frac{1}{3}\sum_{n=0}^{\infty}(z/3)^n.\end{aligned}$$

Therefore,

$$f(z) = \sum_{n=-\infty}^{-1}(-1)^{n+1}\frac{z^n}{2} + \sum_{n=0}^{\infty}\frac{-z^n}{3^n(6)}.$$

6. (15 points) Let $D = \{z = re^{i\theta} : \epsilon < r < R, 0 < \theta < \frac{3\pi}{2}\}$. Express

$$\lim_{\substack{R\to\infty,\\ \epsilon\to 0}} \int_{\partial D} \frac{dz}{1+z^{4/3}}$$

as a sum involving $\int_0^\infty \frac{dx}{1+x^{4/3}}$ and limits of integrals along circular arcs. You do not need to evaluate any integrals or limits, and you do not need to find a numerical value for the integral above.

Solution: Let Γ_R be the 3/4 circle of radius R given by $Re^{i\theta}$ with $0 < \theta < \frac{3\pi}{2}$ and γ_ϵ be the 3/4 circle of radius ϵ. Take $z^{4/3}$ to be the branch defined by the map $re^{i\theta} \mapsto r^{4/3}e^{4i\theta/3}$ where $0 < \theta < 2\pi$. We have that

$$\int_{\partial D} \frac{dz}{1+z^{4/3}} = \int_{\Gamma_R} \frac{dz}{1+z^{4/3}} - \int_{\gamma_\epsilon} \frac{dz}{1+z^{4/3}} + \int_\epsilon^R \frac{dz}{1+z^{4/3}} + \int_{-iR}^{-i\epsilon} \frac{dz}{1+z^{4/3}}.$$

We can rewrite

$$\int_\epsilon^R \frac{dz}{1+z^{4/3}} = \int_\epsilon^R \frac{dx}{1+x^{4/3}}.$$

Also, by parametrizing $z = -ix$,

$$\int_{-iR}^{-i\epsilon} \frac{dz}{1+z^{4/3}} = \int_R^\epsilon \frac{-idx}{1+(xe^{3\pi i/2})^{4/3}} = \int_\epsilon^R \frac{idx}{1+x^{4/3}}.$$

Therefore, if we take the limit as $R \to \infty$ and $\epsilon \to 0$, we have

$$\lim_{\substack{R\to\infty,\\ \epsilon\to 0}} \int_{\partial D} \frac{dz}{1+z^{4/3}} = \lim_{R\to\infty} \int_{\Gamma_R} \frac{dz}{1+z^{4/3}} - \lim_{\epsilon\to 0} \int_{\gamma_\epsilon} \frac{dz}{1+z^{4/3}} + (1+i) \int_0^\infty \frac{dx}{1+x^{4/3}}.$$

7. (10 points) Suppose that $f(z)$ is analytic in a bounded domain D with piecewise smooth boundary, and $f(z)$ extends to be analytic on ∂D. Let $I = \{iy : 0 \leq y < \infty\}$ denote the positive imaginary axis. Suppose that $f(\partial D) \cap I = \emptyset$. Show that $f(z)$ has no zeros in D.

Solution: Suppose that $f(\partial D) \cap I = \emptyset$. In other words, $f(z) \notin I$ for $z \in \partial D$.

We have that if γ is any curve (not necessarily closed) in $\mathbb{C} \setminus I$, then the largest that the change in argument can be is 2π (traveling from just left of I counter-clockwise around the origin to just right of I) while the smallest it can be is -2π (traveling from just right of I clockwise around the origin to just left of I). One way to see this is to define a branch of the argument function $\operatorname{Arg} z$ so that $\frac{\pi}{2} < \operatorname{Arg} z < \frac{5\pi}{2}$, which is continuous on $\mathbb{C} \setminus I$. Then,

$$\int_\gamma d(\arg z) = \operatorname{Arg}(b) - \operatorname{Arg}(a)$$

where a and b are the starting point and ending point of γ, respectively. Then, for this branch of Arg, we can see that $-2\pi < \operatorname{Arg}(b) - \operatorname{Arg}(a) < 2\pi$.

Since $f(z)$ has its image in $\mathbb{C} \setminus I$ for $z \in \partial D$, we can conclude that

$$-2\pi < \int_{\partial D} d(\arg f(z)) < 2\pi.$$

By the argument principle, we know that $\int_{\partial D} d(\arg f(z))$ is always an integer multiple of 2π, so the integral must be 0.

But also, $0 = \int_{\partial D} d(\arg f(z)) = 2\pi(N_0 - N_\infty)$, where N_0 is the number of zeros of $f(z)$ in D, and N_∞ is the number of poles of $f(z)$ in D. Since $f(z)$ is analytic on D, $N_\infty = 0$. Thus, we can conclude that $N_0 = N_\infty = 0$

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Question:	1	2	3	4	5	6	7	Total
Points:	12	18	15	15	15	15	10	100
Score:								