Math 185 Lecture 3

Midterm 2

October 31, 2014

Name: ______________________________

- You will have **50 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ______________________________

1. Determine whether the following statements are true or false. No justification is required.

   (a) (5 points) If $Pdx + Qdy$ is an exact differential, then $\int_\gamma Pdx + Qdy$ is independent of path.

   TRUE false

   (b) (5 points) The function $\sin z$ satisfies $|\sin z| \leq 1$ for all $z \in \mathbb{C}$.

   true **FALSE**

   (c) (5 points) Every harmonic function $u(x, y)$ on a domain D has a harmonic conjugate $v(x, y)$ defined on all of D.

   true **FALSE**

   (d) (5 points) If $\{f_n\}_{n=0}^{\infty}$ is a sequence of analytic functions on D that converges to f, then f is analytic on D.

   true **FALSE**

   (e) (5 points) If $f(z)$ is an analytic function on a domain D, then $f(z)$ has derivatives of all order at every point in D.

   TRUE false

2. (20 points) Suppose that $f(z)$ is an analytic function on the domain $D = \{z : |z| < R\}$, and that

$$f^{(m)}(0) = \begin{cases} (-1)^k & \text{if } 2k = m, \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

Prove that $f(z) = \cos z$ on D.

Solution: Let $g(z) = \cos z$. Then,

$$\begin{aligned} g(0) &= \cos 0 = 1 & & \\ g'(z) &= -\sin z & &\Rightarrow g'(0) = 0 \\ g''(z) &= -\cos z & &\Rightarrow g''(0) = -1 \\ g'''(z) &= \sin z & &\Rightarrow g'''(0) = 0 \\ g^{(4)}(z) &= \cos z & &\Rightarrow g^{(4)}(0) = 1. \end{aligned}$$

Continuing on, we see that

$$g^{(m)}(0) = \begin{cases} (-1)^k & \text{if } 2k = m, \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

This means that $g^{(m)} = f^{(m)}$ for all $m \geq 0$. Since f and g are both analytic on the disk $|z| < R$, by the Corollary on p.146, we have that $g(z) = f(z)$ for all $|z| < R$. Hence, $f(z) = \cos z$ on D.

3. (20 points) Find functions $f_j(z)$ and constants m_j, z_j, c_j such that

$$\int_{|z|=5} \frac{e^z}{(z^2-4)^3(z-6)}dz = \sum_{j=1}^{N} c_j f_j^{(m_j)}(z_j).$$

You do not have to evaluate any of the derivatives $f_j^{(m_j)}(z_j)$.

Solution: Let D_ϵ be the disk of radius 5 with the disks of radius ϵ about ± 2 removed. Then, $f(z) = \frac{e^z}{(z^2-4)^3(z-6)}$ is analytic on D_ϵ, so by Cauchy's theorem,

$$\int_{\partial D_\epsilon} f(z)dz = 0.$$

But we also have that ∂D_ϵ consists of the three circular paths $|z| = 5$, $|z-2| = \epsilon$ and $|z+2| = \epsilon$. If we consider orientation, then

$$\begin{aligned} 0 &= \int_{\partial D_\epsilon} f(z)dz \\ &= \int_{|z|=5} f(z)dz - \int_{|z-2|=\epsilon} f(z)dz - \int_{|z+2|=\epsilon} f(z)dz. \end{aligned}$$

This implies that

$$\int_{|z|=5} \frac{e^z}{(z^2-4)^3(z-6)}dz = \int_{|z-2|=\epsilon} \frac{\frac{e^z}{(z+2)^3(z-6)}}{(z-2)^3}dz + \int_{|z+2|=\epsilon} \frac{\frac{e^z}{(z-2)^3(z-6)}}{(z+2)^3}dz$$

By the Cauchy Integral Formula, the right side becomes

$$\frac{2\pi i}{2!} f_1''(2) + \frac{2\pi i}{2!} f_2''(-2),$$

where $f_1(z) = \frac{e^z}{(z+2)^3(z-6)}$ and $f_2(z) = \frac{e^z}{(z-2)^3(z-6)}$, since $f_1(z)$ is analytic on the disk $|z-2| < \epsilon$ and $f_2(z)$ is analytic on the disk $|z+2| < \epsilon$. So we have $m_1 = m_2 = 2$, $c_1 = c_2 = \pi i$, and $z_1 = 2$, $z_2 = -2$.

4. (15 points) If γ is the straight line path from $-1+i$ to $-1-i$, compute

$$\int_\gamma \frac{1}{z} dz.$$

Solution: Let Log_1 be the branch of the logarithm that takes $z = re^{i\theta}$ to $\log r + i\theta$, in other words, $\text{Log}_1 = \log|z| + i\theta$, where $\theta \in (0, 2\pi)$. Then, Log_1 is analytic on $\mathbb{C}\setminus[0,\infty)$ – in particular, it is analytic along all of γ. We also have that $\frac{d}{dz}\text{Log}_1 z = \frac{1}{z}$, as it differs from $\text{Log}\, z$ by a constant, so $\text{Log}_1 z$ is a primitive for $\frac{1}{z}$. Hence,

$$\int_\gamma \frac{1}{z} dz = \text{Log}_1(-1-i) - \text{Log}_1(-1+i) = \log|-1-i| + i\frac{5\pi}{4} - (\log|-1+i| + i\frac{3\pi}{4}) = \frac{\pi i}{2}.$$

5. Suppose that $u(z)$ is a real harmonic function on $D = \{z \in \mathbb{C} : |z| < 3\}$, and that $u(z) = M$ for z such that $|z| = 2$.

(a) (10 points) Show that $u(0) = M$.

Solution: Since $u(z)$ is harmonic, by the Mean Value Property,

$$u(0) = \int_0^{2\pi} u(2e^{i\theta}) \frac{d\theta}{2\pi} = \int_0^{2\pi} M \frac{d\theta}{2\pi} = M.$$

(b) (10 points) In fact, show that $u(z) = M$ for all $|z| \leq 2$, i.e. $u(z)$ is constant on the disk of radius 2.

Solution: Since $u(z)$ is harmonic on the disk $|z| < 2$ and $|u(z)| = |M|$ for $|z| = 2$ (= boundary of the disk $|z| < 2$), by the Maximum Principle, we have that $|u(z)| \leq |M|$ for all $|z| < 2$ as well.

Then, since $u(z) \leq |M|$ and $|u(0)| = |M|$, by the Strict Maximum Principle, it must be that $u(z) = M$ is constant for all $|z| < 2$ as well. We conclude that for all $|z| \leq 2$, then $u(z) = M$ is constant.

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Question:	1	2	3	4	5	Total
Points:	25	20	20	15	20	100
Score:						