Math 185 Lecture 4

Final Exam

May 14, 2014

Name: ______________________________

- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- You will have **150 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please do not detach the formula sheet from the exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ______________________________

Stereographic projection:

$$x = X/(1-Z) \qquad X = 2x/(|z|^2+1)$$
$$y = Y/(1-Z) \qquad Y = 2y/(|z|^2+1)$$
$$Z = (|z|^2-1)/(|z|^2+1).$$

Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Harmonic conjugate:

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s,y_0)ds + C$$
$$v(B) = \int_A^B -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

Fractional linear transformation:

$$w = f(z) = \frac{z-z_0}{z-z_2}\frac{z_1-z_2}{z_1-z_0}.$$

Mean value property:

$$u(z_0) = \int_0^{2\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi}.$$

Cauchy integral formula:

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}dw.$$

Power series and Laurent series:

$$a_k = \frac{1}{2\pi i}\oint_{|\zeta - z_0|=r} \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta.$$

Residue theorem:

$$\int_{\partial D} f(z)dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}[f(z), z_j].$$

Argument principle:

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi}\int_{\partial D} d\arg(f(z)) = N_0 - N_\infty.$$

Inverse function theorem:

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho} \frac{\zeta f'(\zeta)}{f(\zeta)-w}d\zeta, \qquad |w-w_0| < \delta.$$

1. For each of the following functions, determine whether the given point is a removable singularity, a pole, an essential singularity, or not a singularity. If it is a pole, give the order of the pole.

   (a) (3 points) $z^{-1}\cos\frac{1}{z}$ at $z = 0$

   (b) (3 points) $\frac{1-\cos z}{z^3(z-\pi)}$ at $z = 1$

(c) (3 points) $\frac{(z-3)\sin(\pi z)}{z(z-1)^3}$ at $z = 1$

(d) (3 points) $\frac{z(z-1)^3}{(z-3)\sin(\pi z)}$ at $z = 1$

2. Determine whether the following statements are true or false. No justification is required.

(a) (2 points) If $f(z)$ has an essential singularity at z_0, then $\lim_{z \to z_0} |f(z)| = \infty$.

TRUE **FALSE**

(b) (2 points) If $Pdx + Qdy$ is closed, then $\int Pdx + Qdy$ is path independent.

TRUE **FALSE**

(c) (2 points) If $f(z)$ is analytic at z_0, then $\operatorname{Res}[f(z), z_0] = 0$.

TRUE **FALSE**

(d) (2 points) The gamma function $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ defined for $\operatorname{Re} z > 0$, satisfies $\Gamma(n+1) = n!$ for all positive integers n.

TRUE **FALSE**

(e) (2 points) Every harmonic function on a domain D has a harmonic conjugate on D.

TRUE **FALSE**

(f) (2 points) The number of prime numbers less than x is equal to $\log x$.

TRUE **FALSE**

(g) (2 points) The power series for $\operatorname{Log} z$ centered at $z = -5 + i$ has radius of convergence equal to 1.

TRUE **FALSE**

(h) (2 points) If $f(z)$ is a non-constant function that is analytic on D, then $f(U)$ is open for every open subset $U \subseteq D$.

TRUE **FALSE**

(i) (2 points) The function $p(z) = z^5 + 3z^4 - 11z^3 + 4z + 2$ has 5 roots inside the unit disk $|z| < 1$.

TRUE **FALSE**

3. (15 points) Determine the subset of points in $\mathbb{C}$ for which $f(z) = 2z + \bar{z}^2$ is differentiable.

4. (a) (5 points) Let Γ_R be the semicircle in the upper half-plane of radius R centered at the origin. Show that

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{iz}}{z(z^2+1)} dz = 0.$$

(b) (5 points) Let γ_ϵ be the semicircle in the upper half-plane of radius ϵ centered at the origin. Find

$$\lim_{\epsilon\to 0} \int_{\gamma_\epsilon} \frac{e^{iz}}{z(z^2+1)} dz.$$

(c) (5 points) Compute

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)} dx,$$

by using the residue theorem.

5. (15 points) Find the Laurent series centered at 0 for the function

$$f(z) = \frac{1}{(z+3)(z+1)},$$

that converges at $z = 2$. Find the annulus of convergence for the Laurent series.

6. (15 points) Let $D = \{z = re^{i\theta} : \epsilon < r < R, 0 < \theta < \frac{3\pi}{2}\}$. Express

$$\lim_{\substack{R\to\infty, \\ \epsilon\to 0}} \int_{\partial D} \frac{dz}{1 + z^{4/3}}$$

as a sum involving $\int_0^\infty \frac{dx}{1+x^{4/3}}$ and limits of integrals along circular arcs. You do not need to evaluate any integrals or limits, and you do not need to find a numerical value for the integral above.

7. (10 points) Suppose that $f(z)$ is analytic in a bounded domain D with piecewise smooth boundary, and $f(z)$ extends to be analytic on ∂D. Let $I = \{iy : 0 \leq y < \infty\}$ denote the positive imaginary axis. Suppose that $f(\partial D) \cap I = \emptyset$. Show that $f(z)$ has no zeros in D.

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Question:	1	2	3	4	5	6	7	Total
Points:	12	18	15	15	15	15	10	100
Score:								