1 September 12, 2014

Definition 1.1. $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is *smooth* if it has continuous partial derivatives of all orders.

Need U to be open in order to define derivative. In order to define for arbitrary subsets, need local extension:

Definition 1.2. $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is *smooth* if at each point $x \in X$, there is an open set $x \in U \subset \mathbb{R}^n$ and a smooth map $F : U \to \mathbb{R}^m$ such that $F|_{U \cap X} = f$, i.e. f has a *local* extension in a neighborhood of each point.

Recall open subsets of X in subspace topology are precisely sets of the form $U \cap X$, so smoothness is a "local" property – $f : X \to \mathbb{R}^n$ is smooth if it smooth in a neighborhood of each point of x (in the subspace topology).

Definition 1.3. $f : X \subset \mathbb{R}^n \to Y \subset \mathbb{R}^m$ is a *diffeomorphism* if f and f^{-1} are smooth (and f is a bijection).

examples: circles = knots $\neq$ triangle, etc.

Definition 1.4. A subset $X \subset \mathbb{R}^N$ is a(smooth) *k-dimensional manifold* if it islocally diffeomorphic to $\mathbb{R}^k$, that is every $x \in X$ has a neighborhood (open set) $x \in V \subset X$ that is diffeomorphic to some open set $U \subset \mathbb{R}^k$.

A diffeomorphism $\phi : U \to V$ is a parametrization of (the neighborhood) V.

The inverse diffeomorphism $\phi^{-1} : V \to U$ is called a coordinate system on V.

When ϕ^{-1} is written in coordinates $\phi^{-1} = (x_1, \dots, x_k)$, the smooth functions $x_1, \dots, x_k$ are called coordinate functions.

We say that $\dim X = k$ is the dimension of X.

We can think of U and V as identified by $phi -$ given coordinates $(x_1, \dots, x_k)$ on U, this gives us a point on V by using the parametrization ϕ, and given a point $v \in V$, we can obtain the coordinates "of v" by taking $(x_1(v), \dots, x_k(v))$.

Example: we can show that $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a 1-dimensional manifold.

We can parametrize the set of $(x, y) \in S^1$ on the upper semicircle, i.e. $y > 0$. $\phi_1(x) = (x, \sqrt{1 - x^2})$ takes the open interval $(-1, 1)$ to the upper semicircle with a smooth map. The inverse $\phi_1^{-1}(x, y) = x$ is smooth. For the lower semicircle, we take $\phi_2(x) = (x, -\sqrt{1-x^2})$. This parametrizes all of S^1 except for two points $(\pm 1, 0)$. To parametrize, we can take $\phi_3 = (\sqrt{1-y^2}, y)$ and $\phi_4 = (-\sqrt{1-y^2}, y)$. These four parametrizations cover S^1 and, for each point (at least) one of these gives a local parametrization.

Proposition 1.5. *Given $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ manifolds, $X \times Y$ is a submanifold of $\mathbb{R}^{N+M}$. Also,* $\dim X \times Y = \dim X + \dim Y$.

Proof. Let $k = \dim X$ and $l = \dim Y$. Suppose $(x, y) \in X \times Y$. Then, there exists and open set $W \subset \mathbb{R}^k$ and a local parametrization $\phi : W \to \phi(W) \subset X$, as well as an open subset $U \subset \mathbb{R}^l$ along with a local parametrization $\psi : U \to Y$.

Define $\phi \times \psi : W \times U \subset \mathbb{R}^{k+l} \to X \times Y$ by $\phi \times \psi(w, u) = (\phi(w), \psi(u))$.

One can check that $\phi \times \psi$ is a local parametrization of $X \times Y$ in a neighborhood of (x, y). □

Definition 1.6. If X and Z are both manifolds in $\mathbb{R}^N$ and $Z \subset X$, then Z is a *submanifold* of X.

2 September 15, 2014

Suppose f is a smooth map from (an open set of) $\mathbb{R}^n$ to $\mathbb{R}^m$ and x is in the domain. Then, for any vector $h \in \mathbb{R}^n$, the derivative of f in the direction of h at x is

$$df_x(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

We can define $df_x : \mathbb{R}^n \mathbb{R}^m$ by $h \mapsto df_x(h)$. This map is linear and has matrix form

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Proposition 2.1 (Chain Rule)**.** $d(g \circ f)_x = dg_{f(x)} df_x$.

A linear map can be thought of as its best linear approximation. We can use derivatives to identify the best linear approximation a manifold X at x. Let $\phi : U \to X$ be a local parametrization around x, and assume $\phi(0) = x$ for convenience. Then, the best linear approximation is

$$u \mapsto \phi(0) + d\phi_0(u) = x + d\phi_0(u).$$

Definition 2.2. The *tangent space* of X at x, $T_x(X)$ is the image of the map $d\phi_0 : \mathbb{R}^k \to \mathbb{R}^N$.

$T_x(X)$ is a vector subspace of $\mathbb{R}^N$, and the translation $x + T_x(X)$ is the "best linear approximation" of X at x.

For this to be a real definition, need to make sure it is well-defined. Let $\psi : V \to X$ be another parametrization. Then, by shrinking U and V, can assume that $\phi(U) = \psi(V)$. The map $h : \psi^{-1} \circ \phi : U \to V$ is a diffeomorphism (composition). Then, $\phi = \psi \circ h$, so by chain rule $d\phi_0 = d\psi_0 \circ dh_0$. Hence, the image of $d\phi_0$ is contained in the image of $d\psi_0$. The reverse also holds similarly, so they must be identical.

Proposition 2.3. $\dim T_x(X) = \dim X$.

Proof. Let $\phi : U \to V$ be smooth, and let $\Phi' : W \to U$ be a smooth map that extends ϕ^{-1}. Then $\Phi' \circ \phi = id$, so $d\Phi'_x \circ d\phi = id$. This implies that $d\phi$ must be an isomorphism, i.e. $\dim = k$. □

3 September 17, 2014

For an example of a tangent space, we can take the parametrization of S^1, $\phi : x \mapsto (x, \sqrt{1 - x^2})$.

Then

$$d\phi = \begin{bmatrix} 1 \\ \frac{x}{\sqrt{1-x^2}} \end{bmatrix}.$$

For each point $x \in \mathbb{R}$, this gives a vector that is tangent to S^1 at $(x, \sqrt{1 - x^2})$.

We can do a similar example for S^2, with a parametrization $\phi : (x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$. Then,

$$d\phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{x}{\sqrt{1-x^2-y^2}} & \frac{y}{\sqrt{1-x^2-y^2}} \end{bmatrix}.$$

The column space of the matrix gives a 2-dimensional subspace of $\mathbb{R}^3$ which is tangent to S^2 at $\phi(x, y)$.

Now want to take derivative of $f : X \to Y$.

We take parametrizations $\phi : U \subset \mathbb{R}^k \to \mathbb{R}^N$ and $\psi : V \subset \mathbb{R}^l \to \mathbb{R}^M$. WLOG take $\phi(0) = x, \psi(0) = y$. We have commutative diagram:

$$\begin{array}{ll} X \to (f) & Y \\ U \to (h = \psi^{-1} \circ f \circ \phi) & V \end{array}$$

So we can define $df_x = d\psi_0 \circ dh_0 \circ d\phi_0^{-1}$.

We claim this is independent of parametrization. Indeed, if Ψ and Φ are two different parametrizations of X and Y, and $H = \Psi^{-1} \circ f \circ \Phi$, then $h = \psi^{-1} \circ \Psi \circ \Psi^{-1} \circ f \circ \Phi \circ \Phi^{-1} \circ \phi = \psi^{-1} \circ \Psi \circ H \circ \Phi^{-1} \circ \phi$. Then,

$$d\psi_0 \circ dh_0 \circ d\phi_0^{-1} = d\psi \circ d\psi^{-1} \circ d\Psi \circ dH \circ d\Phi^{-1} \circ \phi \circ d\phi^{-1} = d\Phi \circ dH \circ d\Phi^{-1}.$$

A similar argument shows:

Proposition 3.1 (Chain Rule). *$d(g \circ f)_x = dg_{f(x)} \circ df_x$.*

Proof. Let $\phi : U \to X$, $\psi : V \to Y$ and $\nu : W \to Z$. Then $h = \psi^{-1} \circ f \circ \phi : U \to V$ and $j = \nu^{-1} \circ g \circ \psi : V \to W$, and $j \circ h : U \to W$.

By definition, $d(g \circ f)_x = d\nu_0 \circ d(j \circ h)_0 \circ d\phi_0^{-1}$.

By the chain rule in $\mathbb{R}^N$, $d(j \circ h) = (dj)_0 \circ (dh)_0$, and also $(dj)_0 \circ (dh)_0 = (dj)_0 \circ d\psi_0^{-1} \circ d\psi \circ (dh)_0$. Substituting yields the desired equality. □

4 September 19, 2014

We want to use the tangent space to study smooth manifolds.

Definition 4.1. A function $f : X \to Y$ is a local diffeomorphism if for every $x \in X$, there exists a neighborhood $x \in U$ that maps diffeomorphically to a neighborhood $f(U)$ of $y = f(x)$.

In order to be a local diffeomorphism, note that $df_x : T_x(X) \to T_y(Y)$ must be an isomorphism. This follows from a chain rule argument if $f : U \to V$ is a diffeomorphism.

There is a version of the inverse function theorem for smooth manifolds:

Theorem 4.2 (Inverse Function Theorem). *Suppose that $f : X \to Y$ is a smooth map whose derivative df_x at the point x is an isomorphism (of vector spaces). Then f is a local diffeomorphism at x.*

As an example, $f : \mathbb{R} \to S^1 : t \mapsto (\cos t, \sin t)$ is a local diffeomorphism.

5 September 22, 2014

A local diffeomorphism says that locally, X and Y "look the same". We can make this more precise by noting that we can find local coordinates around x and y so that $f(x_1, \ldots, x_k) = (x_1, \ldots, x_k)$. In particular, if $\phi : U \to X$ is a parametrization around $x \in X$, and df_x is an isomorphism, then the composition $\psi : f \circ \phi : U \to Y$ is a parametrization around $y = f(x) \in Y$ (here it may be necessary to shrink U to a smaller set so that f is a local diffeomorphism).

Definition 5.1. Two maps $f : X \to Y$ and $f' : X' \to Y'$ are equivalent if there exist diffeomorphisms $\alpha : X' \to X$ and $\beta : Y' \to Y$ so that $f \circ \alpha = \beta \circ f'$ (i.e. diagram commutes).

f is locally equivalent to f' at x if there is a local diffeomorphism at x and a local diffeomorphism at $y = f(x)$ that makes the diagram commute.

The inverse function theorem can be interpreted to say that if df_x is an isomorphism, then f is locally isomorphic to the identity.

If $\dim X < \dim Y$, then IFT cannot apply, since the matrix df_x is not square. The closest thing we can hope for is df_x is injective.

Definition 5.2. $f : X \to Y$ is an *immersion* at x if $df_x : T_x(X) \to T_y(Y)$ is injective. If f is an immersion at every point of X, then f is called an immersion.

Example: The canonical immersion is the inclusion map $\mathbb{R}^k \to \mathbb{R}^l$: $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$.

Theorem 5.3 (Local Immersion Theorem)**.** *Suppose that $f : X \to Y$ is an immersion at x and $y = f(x)$. Then there exist local coordinates around x and y such that $f(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0)$. In other words, f is locally equivalent to the canonical immersion near x.*

Proof. We want to give a similar argument to the case when we had a local diffeomorphism. However, we need to modify them amps in order to use the Inverse Function Theorem.

First begin with a local parametrization $\phi : U \to X$ and $psi : V \to Y$ such that $\phi(0) = x$ and $\psi(0) = y$, and write a commutative diagram with $g = \psi^{-1} \circ f \circ \phi : U \to V$ (we may need to shrink U and V to make sure this makes sense).

$dg_0 : \mathbb{R}^k \to \mathbb{R}^l$ is injective, so by changing basis, if necessary, we can assume that dg_0 is a $l \times k$ matrix of the form:

$$\left[\begin{array}{c} I_k \\ \hline 0 \end{array}\right]$$

We augment g to obtain a function $G : U \times \mathbb{R}^{l-k} \to \mathbb{R}^l$: $G(x, z) = g(x) + (0, z)$. Now, G maps an open set of $\mathbb{R}^l$ into $\mathbb{R}^l$ and the matrix of dG_0 is:

$$\begin{array}{c|c} I_k & 0 \\ \hline 0 & I_{l-k} \end{array}$$

In particular, dG_0 is invertible, so we can apply the inverse function theorem, and G is a local diffeomorphism.

ψ and G are both local diffeomorphisms at 0, so $\psi \circ G$ is also a local diffeomorphism at 0. Choosing a small enough open subset $V' \subset U \times \mathbb{R}^{l-k}$, we have that $\psi \circ G : V' \to Y$ is a local parametrization of Y near y. Also, by construction, if h is the canonical immersion, then $g = G \circ h$. Hence, we have that, possibly after shrinking any/all of the open sets,

$$(\psi \circ G) \circ h = \psi \circ g = f \circ \phi$$

so that the diagram commutes, and f is locally equivalent to the canonical immersion. $\square$

Notice that being an immersion is a local property. If X and Y have the same dimension, then $f : X \to Y$ being an immersion means that f is a local diffeomorphism. However, being a diffeomorphism means that f is a local diffeomorphism but also that f is a bijection.

The image of the canonical immersion is a submanifold of Y. For an arbitrary immersion this may not be true. For example, we can map a circle S^1 into $\mathbb{R}^2$ so that it has self intersections. Even if f is injective, we can have problems, as the figure eight can be realized as the image of an immersion from an open interval $(0, 1)$.

On a torus T^2, we can have other problems: let $\mathbb{R}^1 \to T^2$ be any irrational slope (let T^2 be realized by the map $G : \mathbb{R}^2 \to S^1 \times S^1 : (x, y) = (\cos 2\pi x, \sin 2\pi x, \cos 2\pi y, \sin 2\pi y))$. The map G is a local diffeomorphism, so the resulting composition with the map from $\mathbb{R}^1$ into $\mathbb{R}^2$ along a line of irrational slope is an immersion, which is injective. The image of the map is dense in T^2. So in addition to injectivity, we need another condition.

Definition 5.4. A map $f : X \to Y$ is *proper* if the preimage of every compact set is compact.

Definition 5.5. An immersion $f : X \to Y$ is an *embedding* if f is injective and proper.

6 September 24, 2014

Theorem 6.1. *An embedding $f : X \to Y$ maps X diffeomorphically onto a submanifold of Y.*

Proof. To show $f(x)$ is manifold, we want to show that f is an open mapping, so that $f \circ \psi$ is a parametrization whenever ψ parametrizes X.

Let $W \subset X$ be open and suppose $f(W)$ is not open in $f(X)$. Then there exists a sequence of points $y_i \in f(X)$ that do not belong to $f(W)$ but converge to a point $y \in f(W)$. Then the set $C = \{y\} \cup \{y_i\}_{i=1}^{\infty}$ is compact, as it is closed and bounded. Each y_i has exactly one pre-image in X, which we denote x_i so that $f(x_i) = y_i$. We also let $x \in W$ such that $f(x) = y$. By assumption, $f^{-1}(C) = \{x\} \cup \{x_i\}_{i=1}^{\infty}$ is compact. By passing to a subsequence, there exists a subsequence $\{x_{i_j}\}_j$ that converges to some point $z \in X$. By continuity, $f(x_{i_j}) \to f(z)$, but by assumption, $f(x_i) = y_i \to f(x) = y$, so by injectivity of f, it must be that $z = y$. Since $x_{i_j} \to x \in W$, it must be that there are infinitely many x_i that are in W, which contradicts that $y_i \notin f(W)$.

We can conclude that $f(W)$ is a manifold. Since f is a local diffeomorphism from X to $f(X)$ and f is bijective, then f is a diffeomorphism. $\square$

Note that if X is compact, then every map $f : X \to Y$ is proper. Hence, for compact manifolds, every one-to-one immersion is an embedding.

Now we consider when $\dim X > \dim Y$. Then, the best we can hope for for a map $f : X \to Y$ is that $df_x : T_x(X) \to T_y(Y)$ is surjective.

Definition 6.2. Let $f : X \to Y$ be a smooth map between smooth manifolds. Then f is a submersion at x if $df_x : T_x(X) \to T_{f(x)}Y$ is surjective. If f is a submersion for every point $x \in X$, we simply call f a submersion.

For $X = \mathbb{R}^k$ and $Y = \mathbb{R}^l$ with $k > l$, the canonical submersion is the map $(x_1, \dots, x_k) \mapsto (x_1, \dots, x_l)$, which is projection in the first k coordinates.

Theorem 6.3. *Suppose that $f : X \to Y$ is a submersion at x and $y = f(x)$. Then there exist local coordinates around x and y such that $f(x_1, \dots, x_k) - (x_1, \dots, x_l)$. That is, f is locally equivalent to the canonical submersion.*

Proof. Begin with local parametrizations:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi\big\uparrow & & \psi\big\uparrow \\ U & \xrightarrow{g=\psi^{-1}\circ f\circ\phi} & V \end{array}$$

WLOG $\phi(0) = x$, $\psi(0) = y$.

Since dg_0 is surjective, after a change in basis, we can assume it has form

$$I_l \mid 0$$

We define $G(a) = (g(a), a_{l+1}, \ldots, a_k)$ where $a = (a_1, \ldots, a_k)$. Then dG is the identity, so it is a local diffeomorphism at 0. Now $g =$ (canonical submersion)$\circ$ G, so we can write the following:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi\circ G^{-1}\big\uparrow & & \psi\big\uparrow \\ U' \subset \mathbb{R}^k & \xrightarrow{\text{submersion}} & V \end{array}$$

□

As a corollary, we get a condition in which the preimage of a point is a manifold.

Definition 6.4. Let $f : X \to Y$ be a smooth map of manifolds. Then $y \in Y$ is a *regular value* for f if df_x is surjective at every point x such that $f(x) = y$.

Theorem 6.5. *If y is a regular value of $f : X \to Y$, then the preimage $f^{-1}(y)$ is a sub manifold of X, with* $\dim f^{-1}(y) = \dim X - \dim Y$.

Proof. Let y be a regular value of f and suppose that $x \in f^{-1}(y)$. By the local submersion theorem, there exist local coordinates around x and y such that $f(x_1, \ldots, x_k) = (x_1, \ldots, x_l)$.

Hence, in a neighborhood around x, $f^{-1}(y)$ is the set of points $(0, \ldots, 0, x_{l+1}, \ldots, x_k)$. Then, in a small neighborhood of x, we can parametrize $f^{-1}(y)$ by $(x_{l+1}, \ldots, x_k) \mapsto (0, \ldots, 0, x_{l+1}, \ldots, x_k)$. This shows that $f^{-1}(y)$ is a $k - l$ dimensional manifold. □

As an example, this gives us an easy way to show that S^1 is a 1-manifold without having to exhibit parametrizations.

Let $f(x, y) = x^2 + y^2$. Then $df_{(x,y)} = (2x, 2y)$. In particular, $df_{(x,y)}$ is surjective as long as $(x, y) \neq (0, 0)$. So every nonzero real is a regular value of f, and in particular, $S^1 = f^{-1}(1)$ is a 1-manifold.

Similarly, it is easy to see now that $S^n = f_n^{-1}$ is a n-manifold, where $f_n = x_1^2 + \cdots + x_{n+1}^2$. Any other object that we can write as an implicit function can also easily be shown to be a manifold by showing that it is the inverse image of a regular value.

7 September 26, 2014

It is useful to note the following:

Proposition 7.1. *Let Z be the preimage of a regular value y under the smooth map $f : X \to Y$. Then the kernel of the derivative df_x at any point $x \in Z$ is the tangent space to Z, $T_x(Z)$.*

Proof. By definition, f is constant on Z, as $f(x) = y$ for all $x \in Z$. Hence, df_x is zero on $T_x(Z)$, so $T_x(Z)$ is contained in the kernel of df_x. By the regular value assumption, $df_x : T_x(X) \to T_y(Y)$ is surjective, so the dimension of the kernel is

$$\dim T_x(X) - \dim T_y(Y) = \dim X - \dim Y = \dim Z.$$

Hence, $T_x(Z)$ must be the entire kernel of df_x. $\square$

An interesting application is to show that the orthogonal group $O(n)$ is actually a manifold. We think of $n \times n$ matrixes M_n as being $\mathbb{R}^{n^2}$, where with each entry of the matrix being a coordinate in $\mathbb{R}^{n^2}$.

The orthogonal group is defined be the subset of M_n consisting of matrices A such that $AA^T = I$. Note that for any matrix A, AA^T is symmetric, since $(AA^T)^T = AA^T$. The set of symmetric $n \times n$ matrices $S(n)$ can be identified with $\mathbb{R}^{n(n+1)/2}$. So now, we can define a map $f : M(n) \to S(n)$ by $f(A) = AA^T$. Then, $O(n) = f^{-1}(I)$, so it suffices to show that I is a regular value of f (Exercise).

Now, let $g_1, \ldots, g_l$ be smooth, real-value functions on a manifold X of dimension $k \geq l$. We would like to know when the set of common zeroes

$$Z = \{(x_1, \ldots, x_k) : g_1(x_1, \ldots, x_k) = \cdots = g_l(x_1, \ldots, x_k) = 0\}$$

is a manifold. We need to know when 0 is a regular value, i.e. where $g = (g_1, \ldots, g_l)$ is surjective. This happens at $x \in X$ when the linear functions $dg_{1x}, \ldots, dg_{lx}$ are linearly independent on $T_x(X)$. When g_i satisfy this condition, we say they are *independent* at x.

Proposition 7.2. *If $g_1, \ldots, g_l$ are smooth real-value functions on X that are independent at each point where they all vanish, then the set Z of common zeros is a submanifold of X with dimension equal to* $\dim X - l$.

In the situation above, we say that Z is *cut out* by the independent functions $g_1, \ldots, g_l$. A useful definition is as follows:

Definition 7.3. In the situation where Z is a submanifold of X, then the *codimension* of Z in X is $\operatorname{codim} Z = \dim X - \dim Z$.

The converse of the above statement is not always true. Not every submanifold can be cut out by independent functions. But there are some partial converses:

Proposition 7.4. *If y is a regular value of a smooth map $f : X \to Y$, then the preimage submanifold $f^{-1}(y)$ can be cut out by independent functions.*

Proof. Let h be a diffeomorphism of an open subset $W \subset Y$ containing y with an open subset containing the origin in $\mathbb{R}^l$ (i.e. $h = \psi^{-1}$ for some parametrization ψ), where $h(y) = 0$. Let $g = h \circ f$.

Since $h^{-1}(0) = y$ and y is a regular value, then 0 is a regular value of $(h \circ f)^{-1}$. Then the coordinate functions $g_1, \ldots, g_l$ cut out $f^{-1}(y)$. □

Proposition 7.5. *Every submanifold of X is locally cut out by independent functions.*

Proof. Let Z be a submanifold of codimension l, and let $z \in Z$.

Then the inclusion $i : Z \to X$ is an immersion, so by the local immersion theorem, there exists coordinates on Z and X such that $i(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0)$. So Z, locally, is the subset of X such that $x_{k+1} = \cdots = x_{k+l} = 0$. □

We want to generalize the idea of a regular value to a submanifold, i.e. when we have $f : X \to Y$, and $Z \subset Y$ is a sub manifold, then when is $f^{-1}(Z)$ a submanifold of X?

To show that $f^{-1}(Z)$ is a manifold, for each point $x \in f^{-1}(Z)$, we want to show that there exists an open set $x \in U \subset X$ such that $f^{-1} \cap U$ is diffeomorphic to an open subset of $\mathbb{R}^k$, in other words, that $f^{-1} \cap U$ is itself a manifold.

Since Z is a submanifold of Y, in an open set containing y, Z is the zero set of a collection of independent function $g_1, \ldots, g_l$, where $l = \operatorname{codim} Z$ in Y. Near x, then $f^{-1}(Z)$ is the zero set of the function $g_1 \circ f, \ldots, g_l \circ f$. Take $g = (g_1, \ldots, g_l)$, which is a submersion since the g_i are independent.

In order for $f^{-1}(Z)$ to be a manifold, we want 0 to be a regular value of $g \circ f$, i.e. $d(g \circ f) = dg_y \circ df_x$ surjective at any point $x \in f^{-1}(Z)$. Since dg_y is a surjection with kernel $T_y(Z)$, then $d(g \circ f)$ is surjective if and only if

$$\operatorname{Im}(df_x) + T_y(Z) = T_y(Y).$$

Definition 7.6. We say that a smooth map $f : X \to Y$ is transversal to a submanifold $Z \subset Y$, i.e. $f \pitchfork Z$, if

$$\operatorname{Im}(df_x) + T_y(Z) = T_y(Y),$$

for each $x \in f^{-1}(Z)$.

Theorem 7.7. *If $f : X \to Y$ is transversal to a submanifold $Z \subset Y$, then the preimage $f^{-1}(Z)$ is a submanifold of X. Moreover the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y.*

Proof. The above discussion shows that $f^{-1}(Z)$ is a submanifold. Since $f^{-1}(Z)$ and Z are written as the zero set of l independent functions, both have codimension l. □

If we take two submanifolds $X \subset Y$ and $Z \subset Y$, then we can study the intersection $X \cap Z \subset Y$ by using transversality.

If we take the inclusion map $i : X \to Y$, and consider $i^{-1}(Z)$, this gives the intersection $X \cap Z$. In order for $X \cap Z$ to be a manifold, then we need that $i \pitchfork Z$ for each $x \in X \cap Z$. Since $di_x : T_x(X) \to T_x(Y)$ is the inclusion map, then being transversal simply means that

$$T_x(X) + T_x(Z) = T_y(Y).$$

Definition 7.8. Two submanifolds X and Z in Y are transversal in Y, i.e. $X \pitchfork Z$, if

$$T_x(X) + T_x(Z) = T_y(Y).$$

Theorem 7.9. *The intersection of two transversal submanifolds of Y is a submanifold of Y. Moreover,*

$$\operatorname{codim}(X \cap Z) = \operatorname{codim} X + \operatorname{codim} Z.$$

8 September 29, 2014

We want to determine which properties hold, even when manifolds are deformed in a smooth manner.

Definition 8.1. $f_0 : X \to Y$ and $f_1 : X \to Y$ are *homotopic* if there exists a smooth map $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. F is called a *homotopy* between f_0 and f_1, and we write $f_0 \sim f_1$.

The maps $f_t(x) = F(x, t)$ give a family of maps $f_t : X \to Y$ that deform f_0 to f_1.

Exercise: homotopy is an equivalence relation on smooth maps. We call the equivalence class $[f]$, under homotopy, to be the *homotopy class* of f.

Consider intersection of 2 submanifold $X, Z \subset Y$. If X and Z are not transversal, then $X \cap Z$ might still be a manifold, but a small deformation of X and Z can change the intersection. However, if $X \pitchfork Z$, then the intersections are diffeomorphic.

Definition 8.2. We say that a given property of a map (or set of maps) f_0 is *stable* if for any homotopy f_t of f_0, there exists some $\epsilon > 0$ such that for every $0 < t < \epsilon$, f_t also has that property. A collection of maps that possess a particular stable property are called a *stable class* of maps.

Theorem 8.3 (Stability theorem)**.** *The following classes of smooth maps of a compact manifold X into a manifold Y are stable classes:*

(a) local diffeomorphism

(b) immersions

(c) submersion

(d) maps transversal to any specified closed submanifold $Z \subset Y$ (and hence transversal intersection)

(e) *embeddings*

(f) *diffeomorphisms*

We begin with a lemma

Lemma 8.4. *Let K be a compact set. Then any open set U that contains K contains $K_\epsilon = \{x : d(x, s) < \epsilon$ for some $s \in K\}$.*

Proof. Let $\epsilon_s = \sup\{\epsilon : B_\epsilon(s) \subset K\}$. ϵ_s is a continuous function of s, so it achieves its minimum $\epsilon = \min\{\epsilon_s : s \in K\}$ on K. The minimum is nonzero since U is open. □

Proof of Theorem. (b) Let $x \in X$. By injectivity of $d(f_0)_x$, the Jacobian matrix $d(f_0)_x = \left[\frac{\partial (f_0)_i}{\partial x_j}\right]$ has a $k \times k$ submatrix with nonzero determinant. The partial derivatives $\frac{\partial (f_t)_i}{\partial x_j}$ are continuous functions on $X \times I$. The determinant is a continuous function of the partial derivatives, so by composition, it is continuous on $X \times I$. So there exists an open set U_x containing $(x, 0)$ on which the determinant is nonzero, i.e. $d(f_t)$ is injective.

Hence, there exists an open subset of $X \times I$ containing $X \times \{0\}$ on which f_t is an immersion at each point. Since X is compact, by the lemma, this means there exists an $\epsilon > 0$ such that $d(f_t)$ is injective for each point of $X \times [0, \epsilon)$, proving stability.

(a) This follows from (b) because a local diffeomorphism is an immersion where $\dim X = \dim Y$.

(c) The proof is identical, except we find a $l \times l$ submatrix with nonzero determinant.

(d) Transversality is equivalent to the map $g \circ f$ being a submersion at each point for some map g (see the proof that transversality implies that the inverse image is a submanifold), so follows from (c).

(e) We suppose that f_0 is an injection, and show that f_t is injective for small enough t. This suffices since immersion is a stable property, so we know that for small enough t, f_t is an immersion. And since X is compact, any one-to-one immersion is an embedding.

Let $G : X \times [0,1] \to Y \times [0,1]$ be defined by $G(x,t) = (f_t(x), t)$. If f_t is not injective for all t sufficiently small, then there exists a sequence $t_i \to 0$ and $x_i, y_i \in X$ such that $G(x_i, t_i) = G(y_i, t_i)$. Since X is compact, then so is $X \times [0,1]$, so we may find a subsequence of the x_i and y_i that converge. Taking this subsequence and renumbering, if necessary, we assume $x_i \to x$ and $y_i \to y$.

Since f_0 is injective, then

$$G(x,0) = \lim G(x_i, t_i) = \lim G(y_i, t_i) = G(y,0)$$

implies that $x = y$.

We can see that $dG_{(x,0)}$ has the form

$$\begin{array}{c|c} d(f_0)_x & v \\ \hline 0 & 1 \end{array}$$

Since f_0 is injective, so is $d(f_0)_x$. The last column is linearly independent from the left block, so $dG_{(x,0)}$ is also injective. This implies that G is injective in some neighborhood of $(x, 0)$, which contradicts that $G(x_i, t_i) = G(y_i, t_i)$. Thus, f_t must be injective for sufficiently small t.

(f) A diffeomorphism is a surjective embedding, so by stability of embeddings, it suffices to show that f_t is surjective for small enough t.

We first prove the case when X (and hence Y) is connected. Since f_0 is a diffeomorphism, it is a local diffeomorphism, so by stability of local diffeomorphisms, f_t is a local diffeomorphism for small enough t. Hence, $f_t(X)$ is open in Y, as X is open. Moreover, since X is compact, $f_t(X)$ is compact in Y, and hence closed in Y. Since $f_t(X)$ is both open and closed, it must be all of Y, so f_t is surjective for small enough t.

If X has multiple connected components, by compactness, there are finitely many components. So we can use the above argument for each component, then take the smallest ϵ.

$\square$

9 October 6, 2014

We want to know how often we have a regular value so that we can use the Preimage Theorem to obtain a manifold.

Theorem 9.1 (Sard's Theorem)**.** *If $f : X \to Y$ is a smooth map between smooth manifolds, then almost every point of Y is a regular value of f.*

To make precise, we need the notion of having measure zero, which generalizes the idea of having volume zero.

Definition 9.2. A set $A \subset \mathbb{R}^l$ has *measure zero* if it can be covered by countably many rectangular solids with arbitrarily small total volume. In other words, for every $\epsilon > 0$, there exists a countable collection of rectangular solids $\{S_i\}$ such that $A \subset \cup S_i$ and $\sum \mathrm{vol}(S_i) < \epsilon$.

As an example, $\mathbb{Q}$ has measure zero in $\mathbb{R}$. To see this, take an enumeration of the rational numbers, and given $\epsilon > 0$, cover the nth rational number, q_n by a rectangle with volume $\epsilon/2^n$. Then $\sum \epsilon/2^n = \epsilon$.

For a manifold, we define measure zero via parametrizations.

Definition 9.3. A subset $C \subset Y$ has *measure zero* if for every local parametrization ψ of Y, the preimage $\psi^{-1}(Y)$ has measure zero in $\mathbb{R}^l$.

Lemma 9.4. *If $A \subset \mathbb{R}^l$ has measure zero and $g : \mathbb{R}^l \to \mathbb{R}^l$ is a smooth map, then $g(A)$ has measure zero.*

Corollary 9.5. *Having measure zero is independent of parametrization, i.e. C has measure zero if it can be covered by the images of some collection of local parametrizations ψ_α such that $\psi_\alpha^{-1}(C)$ has measure zero for each α.*

Definition 9.6. Let $f : X \to Y$ be a smooth map. A point $y \in Y$ is a *critical value* of f if y is not a regular value.

Theorem 9.7 (Sard's Theorem)**.** *The set of critical values of a smooth map $f : X \to Y$ has measure zero (in Y).*

Notice that no set of measure zero can contain a non-empty open set. If it did, it would contain a ball of radius $r > 0$, for some r, which then contains some rectangular solid with positive volume v. So any set of rectangular solids that cover the set would have to have total volume at least v. As a corollary of Sard's theorem, we have:

Corollary 9.8. *The set of regular values of a smooth map $f : X \to Y$ is dense in Y. If $f_i : X_i \to Y$ are any countable number of smooth maps, then the points of Y that are simultaneously regular values for all of the f_i are dense in Y.*

Proof. The first statement is a restatement of what we already know.

The second statement follows because the countable union of measure zero sets has measure zero. For if C_i has measure zero, then given $\epsilon > 0$, there are rectangular solids that cover C_i so that the total volume is less than $\epsilon/2^i$. Then, the union $\cup C_i$ can be covered by solids that have volume less than $\sum \epsilon/2^i = \epsilon$.

Let C_i be the set of critical values of f_i, which has measure zero. So then $\cup C_i$ also has measure zero. But the complement is the subset of Y such that the points are regular values for all the f_i. $\square$

Definition 9.9. Let $f : X \to Y$ be a smooth map. A point $x\, inX$ is a *regular point* of f if $df_x : T_x(X) \to T_y(Y)$ is surjective. If df_x is not surjective, then x is a *critical point* of f.

Note that this is similar to, but distinct, from the idea of a regular value and a critical value. This distinction is important, as Sard's theorem says that the set of critical values has measure zero. But this is not necessarily true of critical points. If $f : X \to Y$ is the constant map, then every point of X is a critical point, which is not measure zero.

10 October 8, 2014

Implicitly, in the statement of Sard's theorem and the corollary, we discussed the notion of measure zero for a subset of a manifold Y.

Definition 10.1. A subset $C \subset Y$ has *measure zero* if for every local parametrization ψ of Y, the preimage $\psi^{-1}(Y)$ has measure zero in $\mathbb{R}^l$.

Lemma 10.2. *If $A \subset \mathbb{R}^l$ has measure zero and $g : \mathbb{R}^l \to \mathbb{R}^l$ is a smooth map, then $g(A)$ has measure zero.*

Corollary 10.3. *Having measure zero is independent of parametrization, i.e. C has measure zero if it can be covered by the images of some collection of local parametrizations ψ_α such that $\psi_\alpha^{-1}(C)$ has measure zero for each α.*

an interesting question when studying manifolds is given a k-dimensional manifold, how high does N have to be to embed X^k in $\mathbb{R}^N$? e.g. S^2 and T^2 can be embedded in $\mathbb{R}^3$, but the Klein bottle cannot, and can only be embedding in $\mathbb{R}^4$. Whitney's theorem gives an upper bound on how big N must be. In fact, $N = 2k + 1$ is sufficient.

The intuition behind the result is not too difficult to understand: a k-dimensional manifold locally is diffeomorphic to $\mathbb{R}^k$, so we can take k-dimensional pieces and glue them together to obtain X. If at any time when we are gluing them together, we end up with two pieces that need to "occupy the same space," as long as we have $2k+1$ dimensions, we can move them apart, since each only occupies k dimensions of space.

The argument will use Sard's theorem and the tangent bundle. The tangent bundle is the manifold X along with its tangent space at each point.

Definition 10.4. Suppose $X \subset \mathbb{R}^N$ is a k-dimensional manifold. Then the *tangent bundle* of X is

$$T(X) = \{(x, v) \in X \times \mathbb{R}^N : v \in T_x(X)\}.$$

Given a smooth map $f : X \to Y$, we get a map on the tangent bundle $df : T(X) \to T(Y) : (x, v) \mapsto (f(x), df_x(v))$, which we call the global *derivative map*. The derivative map satisfies the chain rule because the point wise derivative map df_x satisfies the chain rule.

Proposition 10.5. *The tangent bundle $T(X)$ of a manifold X is a manifold, and $\dim T(X) = 2 \dim X$.*

Proof. Suppose $x \in X$, and $\phi : U \subset \mathbb{R}^k \to W \subset X$ be a local parametrization. Then, $d\phi : U \times \mathbb{R}^k \to T(W)$ is a parametrization, since $d\phi_y$ is an isomorphism for all y.

This gives a parametrization of any open subset of $T(X)$, which has the form $T(X) \cap W \times \mathbb{R}^N$. □

11 October 10, 2014

Theorem 11.1. *Every k-dimensional manifold admits a one-to-one immersion in $\mathbb{R}^{2k+1}$.*

Proof. We will let $f : X \to \mathbb{R}^N$ be an injective immersion of a k-dimensional manifold. If $N > 2k+1$, then we will show that X can be immersed in $\mathbb{R}^{N-1}$ by projecting onto the orthogonal complement of some vector a, and proceed by induction, starting with the inclusion of X into $\mathbb{R}^N$.

Let $h : X \times X \times \mathbb{R} \to \mathbb{R}^N$ be defined by $h(x, y, t) = t[f(x) - f(y)]$, and $g : T(X) \to \mathbb{R}^N$ by $g(x, v) = df_x(v)$.

Since $\dim(X \times X \times \mathbb{R}) = 2k + 1 < N$, then $dh_{(x,y,t)}$ can never be a surjection. Hence, the only regular values of h are exactly those which are not in the image of h. Similarly, since $\dim T(X) = 2k < N$, the only regular values of g are those not in the image of g.

By Sard's theorem, there exists an $a \in \mathbb{R}^N$ that is a regular value of both h and g, which means that a is not the image of h or g. Since 0 is in the image of both h and g, we can conclude that $a \neq 0$.

Let $H = \{b \in \mathbb{R}^N : b \perp a\}$ be the orthogonal complement of a. Take $\pi : \mathbb{R}^N \to H$ be the orthogonal projection. Consider $\pi \circ f$. Suppose that $\pi \circ f(x) = \pi \circ f(y)$. But then $f(x) - f(y) = ta$ for some $t \in \mathbb{R}$, since they project along a to the same point in H. But then $h(x, y, 1/t) = a$, which contradicts that a is not the image of h. Hence, $\pi \circ f$ is one-to-one.

We also show that $\pi \circ f : X \to H$ is an immersion. Suppose that it is not, so there exists $x \in X$ and nonzero $v \in T_x(X)$ such that $d(\pi \circ f)_x(v) = 0$. Since π is a linear map, then $d(\pi \circ f)_x = \pi \circ df_x$. Hence, $\pi \circ df_x(v) = 0$ implies that $df_x(v) = ta$ for some $t \in \mathbb{R}$. Furthermore, f is an immersion, so $t \neq 0$. Then, $g(x, 1/t) = a$, which contradicts that a is not in the image of g. $\square$

Recall that for a compact manifold X, all maps $f : X \to Y$ are proper, so a one-to-one immersion is in fact an embedding. This proves Whitney's theorem for compact manifolds.

Corollary 11.2. *Let X be a compact k-dimensional manifold. Then X embeds into $\mathbb{R}^{2k+1}$.*

12 October 22, 2014

Up until now, in our definition of a manifold, every point needed to "look" like $\mathbb{R}^k$. In particular, this meant that we never had boundary.

We generalize to manifolds with boundary. In order to allow boundary, we use the model space $H^k = \{(x_1, \ldots, x_k) : x_k \geq 0 =$ upper half space in $\mathbb{R}^k$.

Definition 12.1. A subset $X \subset \mathbb{R}^N$ is a (smooth) k-dimensional manifold with boundary if every point of X has a neighborhood diffeomorphic to an open set in H^k. Such a diffeomorphism is called a local parametrization. The *boundary* of X, denoted ∂X, consists of the points that belong to the image of ∂H^k under some local parametrization. Its complement is called *the interior* of X, denoted $\operatorname{Int}(X) = X - \partial X$.

Recall that smoothness of a function is defined in terms of having a smooth extension onto an open subset. So for a point on ∂X, the fact that $\phi : U \to V \subset X$ is a diffeomorphism implies that it extends on an open set containing the boundary point, i.e. extends to the lower half plane.

Remark: Be careful with the concept of interior and boundary, and do not confuse with the topological notion of interior or boundary of X as a subset of $\mathbb{R}^N$. They will generally not agree, especially when $\dim X < N$.

The product of two manifolds with boundary is generally not a manifold with boundary. This is primarily because the smoothness condition might not be satisfied on boundary. For example, consider $[0,1] \times [0,1]$.

We do have an analogous proposition, however.

Proposition 12.2. *Suppose X is a manifold without boundary and Y is a manifold with boundary. Then $X \times Y$ is a manifold with boundary. Moreover, $\partial(X \times Y) = X \times \partial Y$.*

Proof. Take local parametrizations $\phi : U \subset \mathbb{R}^k \to X$ and $\psi : V \subset H^l \to X$. Then, since $\mathbb{R}^k \times H^l = H^{k+l}$, $\phi \times \psi : U \times V \subset H^l \to X \times Y$ is a parametrization of $X \times Y$. This also shows that $\partial(X \times Y) = X \times \partial Y$. $\square$

We define the tangent space and derivative map using parametrizations as for the manifold without boundary case. For an interior point of a manifold with boundary X, it's easy to see that the notions are the same. Now, if $\phi : U \to X$ is a parametrization ad $x \in \partial U$, then smoothness of ϕ implies that ϕ can be extended to $\Phi : \tilde{U} \to \mathbb{R}^N$ where $\tilde{U}$ is open in $\mathbb{R}^k$ and contains U. Then, $d\phi_u$ is defined to be $d\Phi_u$.

To show that this is well-defined, we meet to check that if $\tilde{\Phi}$ is another extension of ϕ, then we obtain the same derivative. Certainly, $d\Phi_x = d\tilde{\Phi}_x$ for $x \in \operatorname{Int} U$. Now, take $x \to u$, and use continuity of the derivative. This shows that $d\Phi_u = d\tilde{\Phi}_u$. This then implies that we can define the tangent space $T_x(X)$ by $\operatorname{Im} d\phi_u$. Notice that even for a point $x \in \partial X$, the tangent space $T_x(X)$ is a k-dimensional linear subspace – in particular, it contains the whole linear subspace, not just "half" of it.

13 October 24, 2014

For an arbitrary smooth map $f : X \to Y$ between smooth manifolds with boundary, a similar argument applies to df_x by using the same commutative diagram as before.

We also have the following propositions:

Proposition 13.1. *If X is a k-dimensional manifold with boundary, then* $\operatorname{Int} X$ *is a k-dimensional manifold without boundary.*

Proof. For $x \in \operatorname{Int} X$, we can restrict the parametrization to the open upper half-space $\operatorname{Int} H^k$, in which case open subsets of $\operatorname{Int} H^k$ are just open subsets of $\mathbb{R}^k$. $\square$

Proposition 13.2. *If X is a k-dimensional manifold with boundary, then ∂X is a $(k-1)$-dimensional manifold without boundary.*

Proof. The idea is that if $\phi : U \to X$ is a local parametrization of a neighborhood of $x \in \partial X$, then if we take $V = V \cap \partial H^k$, then $\phi|_V : \mathbb{R}^{k-1} \to \partial X$ is a local parametrization of ∂X.

However, recall that in the definition of ∂X, we only needed x to be in the image of ∂H^k for some local parametrization. To make sure that the restriction $\phi|_V$ is in fact a parametrization, we must make sure that if $x \in \partial X$ under one parametrization, then it is with respect to every parametrization. (If it is not, then $\phi|_V(V)$ might not be open in ∂X.)

So suppose that $\phi : U \to X$ and $\psi : W \to X$ are two local parametrizations whose images contain x, and suppose that x is the image of a point ∂U but the image of an interior point of W. Then, $\phi^{-1} \circ \psi : W \to U$ is a diffeomorphism that maps an interior point of W to a boundary point of U. By the inverse function theorem, there is a small open ball around $\psi^{-1}(x)$ that maps to an open ball in U around $\phi^{-1}(x)$, which contradicts that $\phi^{-1}(x)$ is on the boundary of U. $\square$

Since ∂X is a submanifold of X with codimension 1, then the tangent space $T_x(\partial X)$ is a codimension 1 subspace of $T_x(X)$. For a map $f : X \to Y$, we define ∂f to be the restriction of $df_x : T_x(X) \to T_{f(x)}(Y)$ to the subspace $T_x(\partial X)$.

For the generalization of the Pre-image theorem, we will want for a submanifold $Z \subset Y$ and a smooth map $f : X \to Y$ that $f^{-1}(Z)$ is a manifold, but moreover, $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X$.

Theorem 13.3. *Let f be a smooth map of a manifold X with boundary onto a manifold Y without boundary, and suppose $f : X \to Y$ and $\partial f : \partial X \to Y$ are transversal with respect to a submanifold $Z \subset Y$ without boundary. Then $f^{-1}(Z)$ is a manifold with boundary, and $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X$. Moreover, the codimension of $f^{-1}(Z)$ in X is equal to the codimension of Z in Y.*

Proof. That $f^{-1}(Z) \cap \operatorname{Int} X$ is a manifold follows from the transversality condition of $f|_{\operatorname{Int} X}$ with Z. So we only need to check that for $x \in f^{-1}(Z) \cap \partial X$, there is a local parametrization from H^k. The idea is that transversality of ∂f means that $\partial f^{-1}(Z)$ also is a submanifold of codimension l in ∂X. Since ∂X has dimension 1 lower than X, that means that $\partial f^{-1}(Z)$ has 1 dimension lower than $f^{-1}(Z) \cap \operatorname{Int} X$.

Recall that we can write Z as the zero set of some function $g : Y \to \mathbb{R}^{\operatorname{codim} Z}$. Then, $f^{-1}(Z)$ is the zero set of $g \circ f$. Since $x \in \partial X$, there exists a parametrization $\phi : U \subset H^k \to X$ with $\phi(0) = x$. Since $g \circ f \circ \phi$ is smooth, it extends to some function $\tilde{g} : \tilde{U} \subset \mathbb{R}^k \to \mathbb{R}^{\operatorname{codim} Z}$. By transversality, $S = \tilde{g}^{-1}(0)$ is a manifold without boundary in $\mathbb{R}^k$. We want to make sure that $\tilde{g}^{-1}(0) \cap H^k = (g \circ f \circ \phi)^{-1}(0)$ is a manifold with boundary.

But also, by transversality of ∂f, $\tilde{g}^{-1}(0) \cap \partial H^k$ is also a manifold, and thus a submanifold of S with codimension 1 in S. Then, by the local immersion theorem, there is a change in coordinates such that the inclusion of $S \cap \partial H^k$ into S is given by $(x_1, \ldots, x_{k-l-1}) \to (x_1, \ldots, x_{k-l-1}, 0)$, so that $S \cap H^k$ is indeed a manifold with boundary. □

We also have a useful lemma.

Lemma 13.4. *Suppose that S is a manifold without boundary and that $\pi : S \to \mathbb{R}$ is a smooth function with regular value 0. Then the subset $\{s \in S : \pi(s) \geq 0\}$ is a manifold with boundary and the boundary is $\pi^{-1}(0)$.*

Proof. $\pi^{-1}((0, \infty))$ is open in S, so is a submanifold of S with the same dimension as S. If $\pi(s) = 0$, then since 0 is regular, $d\pi_s$ is a submersion at s, so the local submersion theorem says it looks like the canonical submersion $(x_1, \ldots, x_k) = x_k$. So a neighborhood of s in $\pi^{-1}([0, \infty))$ just looks like the subset where $x_k \geq 0$, which is open in H^k. □

We can use this to show that closed unit ball is a manifold with boundary by letting $S = \mathbb{R}^k$ and $\pi(s) = 1 - |s|^2$. The closed unit ball is the set of s such that $\pi(s) \geq 0$.

Sard's theorem is easier.

Theorem 13.5 (Sard's Theorem for manifolds with boundary)**.** *For any smooth map f of a manifold X with boundary into a manifold Y without boundary, almost every point of X is a regular value of both $f : X \to Y$ and $\partial f : \partial X \to Y$.*

Proof. We apply Sard's theorem for $f|_{\operatorname{Int} X}$ and ∂f, which are both smooth maps on manifolds without boundary. So almost every point is a regular value of both.

But if ∂f is regular at x, then so is f, so this implies that almost every value is a regular value of not just $f|_{\operatorname{Int} X}$ and ∂f, but f as well. $\square$

14 October 27, 2014

We will want to prove a classification theorem for one manifolds. In order to do that, we will need a small discussion on Morse functions.

Let $f : \mathbb{R}^k \to \mathbb{R}$ be a smooth function. If $x \in X$ is a critical point of f, then $df_x = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k}) = 0$. We can consider the Hessian matrix $H = (\frac{\partial^2 f}{\partial x_i \partial x_j})$. If H is nonsingular at x, then x is a *non degenerate critical point* of f.

Letting $g = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k})$, the fact that x is a critical point of f means that $g(x) = 0$. Moreover, dg_x is the Hessian, so if H is nonsingular, then g is a local diffeomorphism in a neighborhood of x mapping to a neighborhood of 0. In particular, this means that there can be no other critical points of f in a neighborhood of x, i.e. the critical point are isolated.

Definition 14.1. Let X be a smooth manifold and $f : X \to \mathbb{R}$ be a smooth function, and let x be a critical point of f. Then x is a *non degenerate critical point* if there exists a parametrization ϕ in a neighborhood of x such that $\phi^{-1}(x)$ is a non degenerate critical point for $f \circ \phi$.

Definition 14.2. A function $f : X \to \mathbb{R}$ is a *Morse function* if every critical point of f is nondegenerate.

Theorem 14.3. *Let $f : U \subset \mathbb{R}^k \to \mathbb{R}$ be a smooth function. Then, for almost every $a = (a_1, \dots, a_k) \in \mathbb{R}^k$,*

$$f_a = f + a_1 x_1 + \dots a_k x_k$$

is a Morse function on U.

Proof. Let $g : U \to \mathbb{R}^k$ be defined by

$$g = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k} \right).$$

Then for $p \in U$, we have that

$$(df_a)_p = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_k}\right) + (a_1, \ldots, a_k) = g(p) + a.$$

Hence, p is a critical point of f_a if and only if $g(p) + a = 0$, or in other words, $g(p) = -a$. We also have that the Hessian for f_a at p is dg_p.

By Sard's Theorem, we know that $-a$ is a regular value for g for almost every a. But if $-a$ is a regular value for g, then dg_p is non-singular. This implies that the Hessian for f_a is non-singular. Since every critical point of f_a satisfies $g(p) = a$, then this means that every critical point of f_a is non degenerate. □

Corollary 14.4. *For a smooth function $f : X \to \mathbb{R}$, for almost every $a \in \mathbb{R}^N$, the function $f_a = f + a_1x_1 + \cdots + a_Nx_n$ is a Morse function on X.*

We wish to prove the following:

Theorem 14.5. *Every compact, connected one-dimensional manifold with boundary is diffeomorphic to either a circle or a closed interval.*

We will start by using a Morse function $f : X \to \mathbb{R}$. Let S be the union of the critical points of f and the boundary points of X. Then, since Morse functions have isolated singularities and X is compact, S is finite. We have that $X \setminus S$ is a finite number of connected one-manifolds $L_1, \ldots, L_N$. We will first study the properties of the L_i. Note that f has no critical points on L_i.

Proposition 14.6. *f maps each L_i diffeomorphically onto an open interval in $\mathbb{R}$.*

Proof. Fix an L_i. By assumption, df_x is surjective for each $x \in L_i$, so in particular, is an isomorphism. This implies that $f : L_i \to \mathbb{R}$ is a local diffeomorphism. As L_i is connected, then $f(L_i)$ is connected, and we know that $f(L_i)$ is also open since f is a local diffeomorphism. Hence, $f(L_i)$ is an open interval. As $f(L_i)$ is contained in $f(X)$, which is compact, then $f(L_i)$ must be finite, so $f(L_i) = (a, b)$ for some a, b.

Fix a point $p \in L$ and let $c = f(p)$. Let Q be the set of points d such that there exists a curve $\gamma : [c, d] \to L_i$ such that $f \circ \gamma = Id_{\mathbb{R}}$. Then, Q is open since if we have a path from $q \in Q$ and $\gamma : [c, d] \to L_i$, then for r in an open neighborhood of q, we can extend (or truncate) the path γ to r. Moreover, Q is closed. If $q_j \to q$ and $q_j \in Q$, then we can can take $\gamma = \lim \gamma_j$, where $\gamma_j : [c, d_j] \to L_j$, which we can do by taking a local diffeomorphism in a neighborhood of q. □

15 October 29, 2014

So now X breaks up into a bunch of pieces that are diffeomorphic to bounded open intervals in $\mathbb{R}$, which are glued together pairwise along critical points. By connectedness, we see that we either end up with a circle, or we can end up with at most 2 endpoints that are not glued together, giving us the closed interval. Guillemin and Pollack shows this more explicitly by using a smoothing lemma to modify the map f so that it is also a local diffeomorphism at the critical points. This then builds the explicit diffeomorphism.

Corollary 15.1. *The boundary of any compact 1-dimensional manifold with boundary consists of an even number of points.*

Proof. A compact 1-manifold will consist of finitely many connected components, each of which has 0 or 2 boundary points by the classification theorem for 1-manifolds. □

Theorem 15.2. *If X is a compact manifold with boundary, then there exists no smooth map $g : x \to \partial X$ such that $\partial g : \partial X \to \partial X$ is the identity. That is, there is no "retraction" of X onto its boundary.*

(Non-)retraction theorem. Suppose that such a $g : X \to \partial X$ exists. By Sard's Theorem, there exists a $x \in \partial X$ that is a regular value of g. As x has codimension $\dim \partial X = \dim X - 1$ in ∂X, then $g^{-1}(x)$ has codimension $\dim X - 1$ in X. Hence, $g^{-1}(x)$ is a 1-manifold.

Notice that $\{x\}$ is closed (by Hausdorff), so $g^{-1}(x)$ is closed. As a closed subset of the compact set X, we have that $g^{-1}(x)$ is a compact 1-manifold. By our preimage theorem for manifolds with boundary, and since ∂g is the identity on ∂X,

$$\partial g^{-1}(x) = g^{-1}(x) \cap \partial X = \{x\}.$$

This contradicts our previous corollary. □

Theorem 15.3 (Brouwer Fixed Point Theorem)**.** *Any smooth map $f : B^n \to B^n$ of the closed unit ball to itself must have a fixed point, i.e. $f(x) = x$ for some $x \in B^n$.*

Proof. Suppose that $f : B^n \to B^n$ does not have any fixed points. We will construct a retraction $g : B^n \to \partial B^n$, which contradicts our previous theorem.

Since $f(x) \neq x$, the points $f(x)$ and x determine a straight line. Let $g(x)$ be the point on ∂B^n where the line starting at $f(x)$ passing through x intersects ∂B^n. If $x \in \partial B^n$, then $g(x) = x$, so $\partial g : \partial B^n \to \partial B^n$ is the identity.

If we can show that g is smooth, this would contradict our non retraction theorem. Since $g(x)$ is on the segment from $f(x)$ to x, then $g(x) = t(x - f(x)) + f(x)$ for some $t \geq 0$. As $f(x)$ is smooth, if we can show that t is a smooth function, then $g(x)$ would be a smooth function. Note that $|g(x)|^2 = 1$, and take the above equation and dot product with itself, we yield

$$|g(x)|^2 = 1 = t^2|x - f(x)|^2 + 2tf(x) \cdot (x - f(x)) + |f(x)|^2.$$

This is a quadratic equation in t, so we can solve for t by the quadratic formula and pick the root so that $t \geq 0$.

$$t = \frac{-[f(x) \cdot (x - f(x))] \pm \sqrt{[f(x) \cdot (x - f(x))]^2 - |x - f(x)|^2(|f(x)|^2 - 1)}}{|x - f(x)|^2}.$$

The denominator $|x - f(x)|^2$ is never zero as $f(x) \neq x$, and the two roots of the equation correspond to the two points on the line between $f(x)$ and x that intersects ∂B^n. The discriminant of the square root being zero implies that both of these points are identical, which could only happen if $f(x) = x$.

Thus, if f has no fixed points, then t is a smooth function, so we have a retraction, which contradicts our non retraction theorem. $\square$

16 October 31, 2014

We previously proved that transversality was a stable property for compact domains. We want to now prove that transversality is "generic", even if the domain is not compact. This will not be quite as strong as stability, but will at least tell us that we can deform an arbitrary map to a nearby transversal map.

We generalize the idea of a homotopy to a smooth family of mappings, $f_s : X \to Y$, parametrized by $s \in S$. The smoothness criteria we will requires is that S be a smooth manifold and that $F : X \times S \to Y$ defined by $F(x, s) = f_s(x)$ be smooth. The important theorem is below.

Theorem 16.1. *Suppose that $F : X \times S \to Y$ is a smooth map of manifolds, where only X has boundary, and let Z be any boundary less submanifold of Y. If both F and ∂F are transversal to Z, then for almost every $s \in S$, both f_s and ∂f_s are transversal to Z.*

Proof. The preimage $W = F^{-1}(Z)$ is a submanifold of $X \times S$ with boundary $\partial W = W \cap \partial(X \times S)$ by the transversality condition (preimage theorem).

We will let $\pi : X \times S \to S$ be the projection map and show that whenever s is a regular value for $\pi|_W$, then $f_s \,\bar{\pitchfork}\, Z$, and whenever s is a regular value for $\partial\pi|_{\partial W}$, then $\partial f_s \,\bar{\pitchfork}\, Z$. By Sard's theorem, almost every value of s is a regular value of both, so almost every value $s \in S$ has the right transversality conditions.

Suppose $f_s(x) = z \in Z$. We wish to show $f_s \pitchfork Z$, that is we want

$$\operatorname{Im} d(f_s)_x + T_z(Z) = T_z(Y).$$

By the fact that $F \,\bar{\pitchfork}\, Z$, we know that

$$\operatorname{Im} dF_{(x,s)} + T_z(Z) = T_z(Y).$$

In other words, given $a \in T_z(Y)$, there exists a $b \in T_{(x,s)}(X \times S)$ such that $dF_{(x,s)}(b) - a \in T_z(Z)$. If we want to show that $f_s \,\bar{\pitchfork}\, Z$, then we need to find a $v \in T_x(X)$ such that $d(f_s)_x(v) - a \in T_z(Z)$.

We've previously shown that $T_{(x,s)}(X \times S) = T_x(X) \times T_s(S)$. Thus, we know that $b = (w, e)$, where $w \in T_x(X)$ and $e \in T_s(S)$. If $e = 0$, then we are done, since if we fix s, then $F(x, s) = f_s(x)$, so that $dF_{(x,s)}(w, 0) = df_s(w)$.

The assumption that s is a regular value of π implies that

$$d\pi_{(x,s)} : T_x(W) \times T_s(S) \to T_s(S)$$

is surjective, and we know from the previous exercise (1.2.9) that this is just the projection onto the second factor. Hence, there is a vector $(u, e) \in T_{(x,s)}(W)$, which is in the preimage $d\pi^{-1}(e)$. As F restricted to W maps into Z, then $dF_{(x,s)}(u, e) \in T_z(Z)$. Now, we have that

$$df_s(w-u)-a = dF_{(x,s)}(w-u, 0) = dF_{(x,s)}[(w, e)-(u, e)] = [dF_{(x,s)}(w, e)-a]-dF_{(x,s)}(u, e).$$

Both vectors in the right hand side are in $T_z(Z)$, so the difference is as well. This shows that any vector $a \in T_z(Y)$ can be written as the sum of a vector in the image of $d(f_s)_x$ and in $T_z(Z)$, which is the transversality condition.

A similar argument shows that $\partial f_s \,\bar{\pitchfork}\, Z$. $\square$

As a consequence, the Transversality theorem implies that transversal maps are generic if $Y = \mathbb{R}^M$. If $f : X \to \mathbb{R}^M$ is a smooth map, let S to be (an open ball in) $\mathbb{R}^M$, and define $F : X \times S \to \mathbb{R}^M$ by $F(x, s) = f(x) + s$. Then $F(x, s)$ is a submersion (see the second factor), so is transversal to any submanifold $Z \subset \mathbb{R}^M$. So by the transversality theorem, for almost every s, $f_s(x)$ is transversal to Z.

Now if Y is an arbitrary manifold in $\mathbb{R}^M$. If we extend the codomain to $\mathbb{R}^M$, then we can apply the above argument to find a transversal map into $\mathbb{R}^M$. To then obtain a map into Y, we somehow need to project the map back down to Y.

17 November 3, 2014

We will need the ϵ neighborhood theorem, which we will prove at a later time.

Theorem 17.1 (ϵ neighborhood Theorem)**.** *For a compact boundaryless manifold $Y \subset \mathbb{R}^M$ and $\epsilon > 0$, let $Y^\epsilon = \{w \in \mathbb{R}^M : d(y, w) < \epsilon$ for some $y \in Y\}$ be the ϵ neighborhood of Y. If ϵ is sufficiently small, then each point $w \in Y^\epsilon$ has a unique closest point in Y, denoted $\pi(w)$, and $\pi : Y^\epsilon \to Y$ is a submersion. If Y is not compact, then there still exists a submersion $\pi : Y^\epsilon \to Y$ that is the identity on Y, but ϵ must be allowed to vary as a smooth function on Y, and $Y^\epsilon = \{w \in \mathbb{R}^M : d(w, y) < \epsilon(y)$ for some $y \in Y\}$.*

We first apply the theorem.

Corollary 17.2. *Let $f : X \to Y$ be a smooth map, Y without boundary. Then there is an open ball S in some Euclidean space and a smooth map $F : X \times S \to Y$ such that $F(x, 0) = f(x)$ and for any fixed $x \in X$, the map $s \mapsto F(x, s)$ is a submersion $S \to Y$. In particular, both F and ∂F are submersions.*

Proof. Let $S \subset \mathbb{R}^M$ be the unit ball, and define $F(x, s) = \pi[f(x) + \epsilon(f(x))s]$. As $\pi : Y^\epsilon \to Y$ restricts to the identity on Y, then $F(x, 0) = \pi[f(x) + 0] = f(x)$.

If we fix x, then $s \mapsto f(x) + \epsilon(f(x))s$ is a submersion. A composition of submersions is a submersion, and as π is a submersion, then $s \mapsto F(x, s)$ is a submersion $S \to Y$. Then F and ∂F must also be submersion if we let x vary, as it is for any fixed x. $\square$

Theorem 17.3 (Transversality Homotopy Theorem)**.** *For any smooth map $f : X \to Y$ and any boundary less submanifold Z of the boundary less manifold Y, there exists a smooth map $g : X \to Y$ homotopic to f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$.*

Proof. The previous corollary says that $F(x, s)$ has the property that $F \pitchfork Z$ and $\partial F \pitchfork Z$. The Transversality Theorem then says that for almost every s such that $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$. But f_s is homotopic to f, with the homotopy given by $F(x, ts), t \in [0, 1]$. □

A more general version, which we will not prove now, is given below:

Theorem 17.4 (Extension Theorem)**.** *Suppose that Z is a closed submanifold of Y, both boundary less, and C is a closed subset of X. Let $f : X \to Y$ be a smooth map with $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $C \cap \partial X$. Then, there exists a smooth map $g : X \to Y$ homotopic to f, such that $g \pitchfork Z$, $\partial g \pitchfork Z$, and on a neighborhood of C, we have $g = f$.*

Corollary 17.5. *If, for $f : X \to Y$, the boundary map $\partial f : \partial X \to Y$ is transversal to Z, then there exists a map $g : X \to Y$ homotopic to f such that $\partial g = \partial f$ and $g \pitchfork Z$.*

Proof. This follows if we can show that ∂X is closed in X, by applying the Extension theorem with $C = \partial X$. But this is easy since $\operatorname{Int} X$ is an open subset of X. □

18 November 5, 2014

The previous section said that given an arbitrary map $f : X \to Y$ and a submanifold $Z \subset Y$, we can always deform f by a small homotopy to find a $g \sim f$ such that $g \pitchfork Z$. We will use this to study intersections of manifolds.

Let $X, Z \subset Y$ be submanifolds of Y.

Definition 18.1. X and Z have *complementary dimension* if $\dim X + \dim Z = \dim Y$.

In the case that $X \pitchfork Z$, then we have that $\operatorname{codim} X \cap Z = \operatorname{codim} X + \operatorname{codim} Z = \dim Z + \dim X = \dim Y$. In other words, $X \cap Z$ is a 0-dimensional manifold.

If we further assume that X is compact and Z is closed, then $X \cap Z$ is a compact 0-manifold, so it is a finite number of points. We want to create an invariant that counts the number of points in the intersection, $\#(X \cap Z)$.

Now, if we have arbitrary X and Z such that X is compact and Z is closed, removing the assumption that $X \pitchfork Z$, then the previous machinery we developed of the Transversality Homotopy Theorem says that we can deform it a little bit to make the intersection transversal. More formally, we can take the inclusion map $i_0 : X \to Y$, and there is a map $i_1 : X \to Y$ homotopic to i_0 and $i_1 \pitchfork Z$. Instead of thinking of homotoping i_0 via the family i_t, we can think of "deforming" X, by taking $X_t = i_t(X)$ to be the image of X as we "deform" i_t. Now, $X_1 \pitchfork Z$.

Unfortunately, different deformations of X may give a different number of points in the intersection $X \cap Z$. Luckily, we can still salvage the idea by taking the number of intersections mod 2. It turns out that although the number of intersection points may change, it will always be even, or it will always be odd.

(draw example of two circles that are tangent − deforming either gives 0 intersection or 2 intersection)

Instead of using only the inclusion map, we can generalize the situation by starting with a compact manifold X and a smooth map $f : X \to Y$ transversal to a closed submanifold $Z \subset Y$, where $\dim X + \dim Z = \dim Y$. Then, $f^{-1}(Z)$ is closed and has codimension in X equal to the codimension of Z in Y. But the codimension of Z in Y is X, so $f^{-1}(Z)$ is 0-dimensional, so it is a finite set.

Definition 18.2. In the above situation where $f \pitchfork Z$, the *mod 2 intersection number* of f with Z is $I_2(f, Z) = \#f^{-1}(Z) \pmod 2$.

Now we will need the following theorem to define the mod 2 intersection number for an arbitrary smooth map.

Theorem 18.3. *If $f_0, f_1 : X \to Y$ are homotopic and both transversal to Z, then $I_2(f_0, Z) = I_2(f_1, Z)$.*

Proof. Let $F : X \times I \to Y$ be a homotopy of f_0 and f_1. Notice that $X \times \{0,1\} = \partial(X \times I)$ is closed, so by the Extension Theorem, we may assume that $F \pitchfork Z$ (if not, there is a $G \sim F$ such that $G = F$ on $X \times \{0,1\}$, i.e. $G(x,0) = F(x,0) = f_0(x)$ and $G(x,1) = F(x,1) = f_1(x)$).

Also, since $f_0 \pitchfork Z$ and $f_1 \pitchfork Z$, we have that $\partial F : \partial(X \times I) = X \times \{0,1\}$ is transversal to Z.

This means that $F^{-1}(Z)$ is a one dimensional submanifold of $X \times I$, as $\dim(X \times I) = \dim X + 1$ and X and Z have complementary dimension. We also have that

$$\partial F^{-1}(Z) = F^{-1}(Z) \cap \partial(X \times I) = f_0^{-1}(Z) \times \{0\} \cup f_1^{-1}(Z) \times \{1\}.$$

From the classification of 1-manifolds, we know that $\partial F^{-1}(Z)$ must have an even number of points. But

$$\#\partial F^{-1}(Z) = \#f_0^{-1}(Z) + \#f_1^{-1}(Z),$$

so this means that

$$I_2(f_0, Z) = \#f_0^{-1}(Z) = \#f_1^{-1}(Z) = I_2(f_1, Z) \pmod 2.$$

$\square$

This says that given an arbitrary map $f : X \to Y$, even if it is not transversal to Z, we can still define the mod 2 intersection number.

Definition 18.4. Let $g : X \to Y$ be smooth, with $Z \subset Y$ a submanifold that such that X and Z have complementary dimension. Then we define the *mod 2 intersection number* of g by finding a homotopic map f such that $f \pitchfork Z$, and define $I_2(g, Z) = I_2(f, Z)$. By the previous theorem, $I_2(g, Z)$ is well-defined, it does not depend on the choice of the homotopic map f.

By transitivity of homotopy, we also have

Corollary 18.5. *If $g_0, g_1 : X \to Y$ are arbitrary smooth maps that are homotopic, then $I_2(g_0, Z) = I_2(g_1, Z)$.*

Now, if we let X also be a submanifold of Y and replace f by the inclusion map, we can define the *mod 2 intersection number* of X with Z by $I_2(X, Z) = I_2(i, Z)$ where $i : X \to Y$ is the inclusion.

If $I_2(X, Z) = 1$, then X and Z cannot be pulled apart, no matter how we deform X, e.g. two curves in a plane, or two curves on T^2.

In the case where $\dim X = \frac{1}{2} \dim Y$, notice that X has complementary dimension with itself in Y. So we can consider $I_2(X, X)$, the *mod 2 self-intersection number* of X. A good example is a curve on the Mobius strip. Convince yourself that $I_2(X, X) = 1$, and that no matter how you deform X, you cannot separate it from the original position of X.

19 November 7, 2014

If X is the boundary of some submanifold $W \subset Y$, then $I_2(X, Z) = 0$. The idea is that if $X \pitchfork Z$, then each time Z "enters" X, then it also must "exit" X, otherwise Z cannot close up.

Theorem 19.1 (Boundary Theorem)**.** *Suppose that X is the boundary of some compact manifold W and $g : X \to Y$ is a smooth map. If g may be extended to all of W, then $I_2(g, Z) = 0$ for any closed submanifold Z in Y of complementary dimension.*

Proof. Let $G : W \to Y$ extend g to W, such that $\partial G = g$. By the Transversality Homotopy Theorem, we can find $F : W \to Y$ homotopic to G such that $F \pitchfork Z$ and $\partial F \pitchfork Z$. Let $f = \partial F$, then we have that $f \sim g$, so that $I_2(g, Z) = I_2(f, Z) = \#f^{-1}(Z)$. But $F^{-1}(Z)$ is a compact one-dimensional manifold with boundary, so $\partial F^{-1}(Z) = f^{-1}(Z)$ is even. $\square$

Intersection theory can also be used to define an invariant for maps that is preserved under homotopy.

Theorem 19.2. *If $f : X \to Y$ is a smooth map of a compact manifold X into a connected manifold Y and $\dim X = \dim Y$, then $I_2(f, \{y\})$ is the same for all $y \in Y$.*

Proof. Given any $y \in Y$, we have by the Transversality Homotopy Theorem that there is a map homotopic to f that is transversal to $\{y\}$. So WLOG, we assume that $f \pitchfork \{y\}$. By the "Stack of Records Theorem" (Chapter 1, Section 4, Exercise 7 on Homework #4), we have that there is an open neighborhood U of y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_n$ of finitely many open sets, each diffeomorphic mapped to U by f.

Hence, $I_2(f, \{z\}) = n$ for al $z \in U$, so that $I_2(f, \{z\}$ is locally constant. As Y is connected, then it must be that $I_2(f, \{z\})$ is constant for all $z \in Y$ $\square$

This gives rise to the definition below:

Definition 19.3. Let $f : X \to Y$ be a smooth map of a compact manifold X to a connected manifold Y with $\dim X = \dim Y$. Then, $\deg_2(f) = I_2(f, \{y\}), y \in Y$ is called the *mod 2 degree* of f.

We know that intersection number is the same for homotopic maps, so we immediately get that

Theorem 19.4. *If $f \sim g$, then* $\deg_2(f) = \deg_2(g)$.

As an immediate corollary of the Boundary Theorem, we get that

Corollary 19.5. *If $X = \partial W$ and $f : X \to Y$ may be extended to all of W, then* $\deg_2(f) = 0$.

We can apply the degree of a map to determine with a function has a zero within a given set. Let $p : \mathbb{C} \to \mathbb{C}$ be a smooth function (here, we use smooth to mean that it is a smooth map $\mathbb{R}^2 \to \mathbb{R}^2$, where $\mathbb{C}$ is considered to be $\mathbb{R}^2$). Let W be a smooth compact region in the plane. We want to know if $p(z) = 0$ for $z \in W$. Assume first that $p(z)$ has no zeros on ∂W, so that $f(z) = \frac{p(z)}{|p(z)|} : \partial W \to S^1$ is defined and is smooth as a map of compact one-manifolds.

If $p(z)$ has no zeros inside W, then we have that $f(z)$ is defined on all of W, so the previous theorem says that $\deg_2(f) = 0$. Hence,

Corollary 19.6. *If* $\deg_2(\frac{p(z)}{|p(z)|}) \neq 0$, *then $p(z)$ has a zero inside W.*

One can think of this as a mod 2 argument principle. We obtain as a further corollary:

Corollary 19.7. *If $p(z)$ is a polynomial of odd degree, then $p(z)$ has a root.*

Proof. Let $p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$ be a polynomial of odd degree (i.e. m is odd). Define a homotopy

$$p_t(z) = tp(z) + (1-t)z^m = z^m + t(a_1 z^{m-1} + \cdots + a_m).$$

Then, $p_0(z) = z^m$, and $p_1(z) = p(z)$.

For W a ball of sufficiently large radius, $p_t(z)$ has no zeros in ∂W (as $|z| \to \infty$, we can easily see that $|p_t(z)| \to \infty$, so $p_t(z)$ can't be 0).

So then $\frac{p_t(z)}{|p_t(z)|}$ is a homotopy between $\frac{p_0}{|p_0|}$ and $\frac{p_1}{|p_1|} = \frac{p}{|p|}$, so they have the same mod 2 degree. But $\frac{p_0}{|p_0|} = \frac{z^m}{|z^m|}$. On ∂W = boundary of a large disk of radius R, the denominator is a just a constant R^m, and we see that for any point $y \in S^1$, then $\frac{z^m}{R^m}$ has exactly m pre images in ∂W. Thus, $\deg_2(\frac{p_0}{|p_0|}) = \deg_2(\frac{p}{|p|}) = m \bmod 2$. So if m is odd, then p must have a zero inside W. $\square$

20 November 10, 2014: Jordan-Brouwer Separation Theorem

The Jordan Curve Theorem is a classical theorem that says that a simple closed curve in $\mathbb{R}^2$ divides the plane into two pieces – the "inside" of the curve and the "outside" of the curve.

Our goal will be to prove this separation theorem in the general case for an $n-1$ dimensional manifold embedded in $\mathbb{R}^n$. The statement we will prove is

Theorem 20.1. *Let X be a compact, connected manifold of codimension 1 in $\mathbb{R}^N$ (we refer to X as a hyper surface). The complement of X in $\mathbb{R}^N$ consists of two connected open sets, the "outside" D_0 and and "inside" D_1. Moreover, $\bar{D_1}$ is a compact manifold with boundary $\partial\bar{D_1} = X$.*

Give some example, starting with a curve in $\mathbb{R}^1$, then maybe some surfaces like S^2 or T^2 in $\mathbb{R}^3$.

Consider a compact, connected $n-1$ dimensional smooth manifold X and a smooth map $f : X \to \mathbb{R}^n$. We want to know how f wraps the image of X around in $\mathbb{R}^n$. Fix a point $z \in \mathbb{R}^n \setminus f(X)$, and consider the unit vector

$$u(x) = \frac{f(x) - z}{|f(x) - z|}.$$

The quantity $u(x)$ indicates the direction from z to $f(x)$. From our study of mod 2 intersection theory, we know that $u : X \to S^{n-1}$ hits almost every direction the same number of times mod 2, which we previously called $\deg_2(u)$. For a curve in the plane, we can see that this measures how f winds the image of $X = S^1$ around z. Let's pick an $x_0 \in S^1$ and denote $v = u(x_0)$. Then as the image of f winds around, $u(x)$ will return to v each time that we make a full circuit around z. So we will make this into a general definition:

Definition 20.2. The *mod 2 winding number* of f around z is $W_2(f, z) := \deg_2(u)$.

The exercises in the book prove this theorem (sketch provided below):

We first start with a proof of the theorem below.

Theorem 20.3. *Suppose that X is a $n-1$ dimensional manifold that is the boundary of a compact manifold with boundary, D, and let $F : D \to \mathbb{R}^n$ be* *a smooth map extending $f : X \to \mathbb{R}^n$, i.e. $\partial F = f$. Then $F^{-1}(z)$ is a finite set, and $W_2(f, z) = \# F^{-1}(z)$ mod 2. In other words, f winds X around z as often as F hits z, mod 2.*

1. **Show that if F does not hit z, then $W_2(f, z) = 0$**

   From a theorem on p. 81, we have that if $X = \partial D$ and $u : X \to Y$ can be extended to all of D, then $\deg_2(f) = 0$. Since $u(x) = \frac{f(x)-z}{|f(x)-z|}$ and F extends f to D, then we can extend u to D by $U(x) = \frac{F(x)-z}{|F(x)-z|}$. This is a smooth map because by assumption, F does not hit z, so the denominator is never 0. Hence, by the theorem, $W_2(f, z) = \deg_2(u) = 0$.

2. **Suppose that $F^{-1}(z) = \{y_1, \ldots, y_l\}$, and around each point y_i let B_i be a ball (that is B_i is the image of a ball in $\mathbb{R}^n$ via some local parametrization of D). Demand that the balls be disjoint from one another and from $X = \partial D$. Let $f_i : \partial D_i \to \mathbb{R}^n$ be the restriction of F, and prove that**

   $$W_2(f, z) = W_2(f_1, z) + \cdots + W_2(f_l, z) \textbf{ mod 2}.$$

   Take
   $$D' = D \setminus (\cup_{i=1}^{l} B_i).$$
   Then $X \cup (\cup_{i=1}^{l} \partial B_i) = \partial D'$. For our extension F, we have that $F|_{D'}$ does not hit z, and apply the previous exercise. This gives

   $$W_2(f, z) + W_2(f_1, z) + \cdots + W_2(f_l, z) = 0 \text{ mod } 2,$$

   which proves the statement.

3. **Use the regularity of z to choose the balls B_i so that $W_2(f_i, z) = 1$, and thus prove the theorem.**

   Suppose $y_i \in F^{-1}(z)$. As z is a regular value of F, and F maps a n-dimensional manifold D into $\mathbb{R}^n$, this must mean that dF_{y_i} is an isomorphism, which implies that F is a local diffeomorphism in a neighborhood of y_i. So choose the B_i small enough such that $F|_{B_i}$ is diffeomorphic (via F) to a small sphere near z. Then $u_i(x) = \frac{f_i(x)-z}{|f_i(x)-z|}$ hits every vector exactly once, so $W_2(f_i, z) = \deg_2 u_i = 1$.

We can apply this theorem in the case where X is actually a smooth manifold of dimension $n-1$ in $\mathbb{R}^n$. If X separates $\mathbb{R}^n$ into the "inside" and "outside", then X is the boundary of a compact n-dimensional manifold with boundary D, which is the inside of X. If we take f to be the inclusion map on X and F to be the extension to the inclusion on D, then we can use $W_2(X,z) := W_2(i,z)$ to determine whether $z \in D$ is on the inside (in which case $W_2(X,z) = 1$) or $z \notin D$ is on the outside (in which case $W_2(X,z) = 0$).

This gives the "converse" of the separation theorem.

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Now we show that we can use the winding number to find the inside and outside.

4. **Let $z \in \mathbb{R}^n \setminus X$. Prove that if x is any point of X and U any neighborhood of x in $\mathbb{R}^n$, then there exists a point of U that may be joined to z by a curve not intersecting X.**

   Let A be the subset of X for which the above is true. A is closed because if $x_i \to x$ and each $x_i \in A$, then any open neighborhood U of x contains at least one x_i. Then, U is also a neighborhood of x_i, and as $x_i \in A$, there is a point in U that may be joined to z by a curve not intersecting X.

   To show A is open, let $x \in A$. Then, $X \to \mathbb{R}^n$ is an immersion, so by the local immersion theorem, there are local coordinates around x so that in an open neighborhood U of x, X is $\mathbb{R}^{n-1}$ inside $\mathbb{R}^n$. As $x \in A$, there is a point $x' \in U$ that is connected to z by a path. Then, for any other point $y \in X \cap U$, we can connect y to x', then concatenate with x' to z to get a path (this might not be a smooth path, but in this case, it's OK if it isn't).

   So A is either empty or all of X. But A is non-empty since we can take a ray from z that intersects X. Let x be the first point on the ray that intersects X. Then $x \in A$.

5. **Show that $\mathbb{R}^n \setminus X$ has, at most, two connected components.**

   Again, using the local immersion theorem at a point $x \in X$, there is a neighborhood U of x where the inclusion of X into $\mathbb{R}^n$ is locally equivalent to the the canonical immersion $(x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, 0)$.

Without loss of generality, we let x map to 0. Then, we can take a small ball $B_\epsilon(0)$. Now, $B_\epsilon(0) \setminus X$ has two components, the upper half sphere and lower half sphere. Take z_0, z_1 in each component. WLOG, we also shrink U so that it is diffeomorphic to B_ϵ.

Now let $y \in \mathbb{R}^n \setminus X$ be an arbitrary point. By the previous exercise, there exists a point in U that is connected to y through a path not intersecting X. But each point in U is connected to either z_0 or z_1 by a path not intersecting X.

Hence, $\mathbb{R}^n \setminus X$ has at most 2 components, one containing z_0 and the other z_1.

6. **Show that if z_0 and z_1 belong to the same connected component of $\mathbb{R}^n \setminus X$, then $W_2(X, z_0) = W_2(X, z_1)$.**

   If z_0 and z_1 are in the same connected component, then there is a curve $\gamma : [0,1] \to \mathbb{R}^n \setminus X$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_1$. Moreover, $\gamma(t) \notin X$ for any t. So we can see that

$$u_t(x) = \frac{x - \gamma(t)}{|x - \gamma(t)|}$$

   is a well-defined homotopy between $u_0(x) = \frac{x - z_0}{|x - z_0|}$ and $u_1(x) = \frac{x - z_1}{|x - z_1|}$. Homotopic maps have the same degree, so $W_2(X, z_0) = W_2(X, z_1)$.

7. **Given a point $z \in \mathbb{R}^n \setminus X$ and a direction vector $v \in S^{n-1}$, consider the ray r emanating from z in the direction of v,**

$$r = \{z + tv : t \geq 0\}.$$

   Check that the ray r is transversal to X if and only if v is a regular value of the direction map $u : X \to S^{n-1}$. In particular, almost every ray from z intersects X transversally.

   We have from a previous exercise (Chapter 1, Section 5, Exercise 7) that $g \circ f \pitchfork W$ if and only if $f \pitchfork g^{-1}(W)$.

   Let f be the inclusion $i : X \to \mathbb{R}^n$ and $g : \mathbb{R}^n \setminus \{z\} \to S^{n-1}$ given by $g(y) = \frac{y - z}{|y - z|}$. Then, $u(x) = g \circ f = g \circ i$. The condition of v being a regular value of u is the same thing as u being transversal to $\{v\}$.

We have that $u \pitchfork \{v\}$ and if only if $i \pitchfork g^{-1}(\{v\})$. But as i is the inclusion, the image of di_x is just $T_x(X)$. And $g^{-1}(\{v\}$ is the set of points in $\mathbb{R}^n \setminus \{z\}$ on the ray from z in the direction of v.

Hence, the latter transversality condition is equivalent to saying that $X \pitchfork r$, as desired.

8. **Suppose that r is a ray emanating from z_0 that intersects X transversally in a nonempty (necessarily finite) set. Suppose that z_1 is any other point on r (but no on X), and l be the number of times r intersects X between z_0 and z_1. Verify that $W_2(X, z_0) = W_2(X, z_1) + l$ mod 2.**

   By the assumption that the intersection of r with X is transversal, Exercise 7 states that v is a regular value of u for z_0. But r can also be considered as a ray from z_1, so the ray emanating from z_1 also intersects X transversally, and hence v is a regular value of u for z_1. But $W_2(X, z_0)$ is the mod 2 degree of u, which is the number of points in the preimage of v, which we can also count as the number of times the ray from z_0 intersects X. Similarly, $W_2(X, z_1)$ is the number of times the ray from z_1 intersects X, so $W_2(X, z_1) + l = W_2(X, z_0)$.

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9. **Conclude that $\mathbb{R}^n \setminus X$ has precisely two components, $D_0 = \{z : W_2(X, z) = 0\}$ and $D_1 = \{z : W_2(X, z) = 1\}$.**

   Fix a $z_0 \in \mathbb{R}^n \setminus X$. WLOG, assume $z_0 \in D_0$. Then, since almost every ray r from z_0 is transversal, take any ray from z_0 that intersects X transversally. Take z_1 to be any point on the ray after the first intersection with X. By the Exercise 8, then we must have that $W_2(X, z_1) = W_2(X, z_0) + 1 = 1$. Hence, $z_1 \in D_1$.

   By Exercise 5, $\mathbb{R}^n \setminus X$ has at most two connected components. By Exercise 6, any two points in the same connected component have the same mod 2 winding number. Thus, D_0 and D_1 make up the two components of $\mathbb{R}^n \setminus X$.

10. **Show that if z is very large, then $W_2(X, z) = 0$.**

As X is compact, X is bounded, and there exists a ball $B_r(0)$ such that $X \subset B_r(0)$. Now, take a point z such that $|z| = 2r$.

Then, we can see that the image of u with respect to this z is contained in $u(\bar{B}_r(0))$. But we can see that then the image of u is contained in an area of S^{n-1} whose angular diameter is $\frac{\pi}{3}$ (draw a picture). In other words, then there is a non measure 0 set of v for which $\#u^{-1}(v) = 0$, which implies that $W_2(X, z) = \deg_2 u = 0$.

11. **Prove the Jordan-Brouwer Separation Theorem: The complement of the compact connected hyper surface (i.e. codimension 1 submanifold) X in $\mathbb{R}^n$ consists of two connected open sets, the "outside" D_0 and the "inside" D_1. Moreover, $\bar{D}_1$ is a compact manifold with boundary $\partial\bar{D}_1 = X$.**

    By exercise 10, if $|z|$ is sufficiently large, then $W_2(X, z) = 0$. Hence, the set $D_0 = \{z : W_2(X, z) = 1\}$ must be bounded. So then by Heine-Borel, $\bar{D_0}$ is closed and bounded, which implies compact.

    That D_0 and D_1 are open follows because X is compact, hence closed. Thus, $\mathbb{R}^n \setminus X$ is open, so each component of $\mathbb{R}^n \setminus X$ is open. The open condition also means that D_0 and D_1 are n dimensional manifolds, as every point x of D_i has an open ball around x that is contained wholly inside D_i.

    So now, it suffices to show that every point $x \in \partial\bar{D}_1 = X$ has a local parametrization with an open subset of H^n. The inclusion $i : X \to \mathbb{R}^n$ is an immersion, so by the local immersion theorem, there exists a parametrization ϕ taking an open subset $U \subset \mathbb{R}^n$ diffeomorphically to a neighborhood $V \subset \mathbb{R}^n$ of x, such that $X \cap V = \phi(U \cap \mathbb{R}^{n-1})$. Then, either $\phi(H^n \cap U)$ or $\phi(-H^n \cap U)$ parametrizes a neighborhood of x in D_1.

12. **Given $z \in \mathbb{R}^n \setminus X$, let r be any ray emanating from z that is transversal to X. Show that z is inside X if and only if r intersects X in an odd number of points.**

    Suppose $|z|$ is very large so that z is outside of X. Then, by Exercise 10, we know that $W_2(X, z) = 0$, so r intersects X an even number of times.

    Conversely, if r intersects X an even number of times, then pick a z_1 on the ray r such that $|z_1|$ is very large and z_1 is outside of X. Then, the part of the ray starting from z_1 does not intersect X, so the number of intersections between z and z_1 is even, so $z, z_1 \in D_0$. As z is contained in the same component as z_1, which is outside of X, then z is also outside of X.

23 November 17, 2014

We will use the Jordan-Brouwer Separation Theorem and the mod 2 intersection theory to show that S^n and $T^n = S^1 \times \cdots \times S^1$ are not diffeomorphic.

We will need the boundary theorem.

Theorem 23.1 (Boundary Theorem)**.** *Suppose that X is the boundary of some compact manifold W and $g : X \to Y$ is a smooth map. If g may be extended to all of W, then $I_2(g, Z) = 0$ for any closed submanifold Z in Y of complementary dimension.*

Corollary 23.2. *Suppose that X is the boundary of some compact manifold W and Z is a submanifold of complementary dimension in Y. Then, $I_2(X, Z) = 0$.*

Proof. This follows immediately from the boundary theorem by taking the inclusion map $i_X : X \to Y$, which then extends smoothly to $i_W : W \to Y$. Since $I_2(X, Z)$ is defined to be $I_2(i_X, Z)$, then the boundary theorem says that $I_2(X, Z) = I_2(i_X, Z) = 0$. □

We then arrive at the following result from the Jordan-Brouwer Separation Theorem:

Lemma 23.3. *Suppose that X, Z are manifolds in $\mathbb{R}^2$, and $X \cong Z \cong S^1$. Then, $I_2(X, Z) = 0$.*

Proof. X is a manifold of codimension 1 in $\mathbb{R}^2$, so by the Jordan-Brouwer Separation Theorem (Jordan Curve Theorem), X is the boundary of a compact manifold W. By the previous corollary, it follows that $I_2(X, Z) = 0$. □

Lemma 23.4. *Suppose that $X, Z \subset Y$ are submanifolds, and $f : Y \to Y'$ is a diffeomorphism. Then, $I_2(X, Z) = I_2(f(X), f(Z))$.*

Proof. Since a diffeomorphism f is a bijection, f preserves the intersection, as a set, so it preserves the number of points in the intersection. So it suffices to show that if $X \pitchfork Z$, then $f(X) \pitchfork f(Z)$. But this follows easily, as a diffeomorphism f induces an isomorphism df_x on tangent spaces.

In particular, if $X \pitchfork Z$, then this implies that $T_x(X) + T_x(Z) = T_x(Y)$ for all $x \in X \cap Z$.

Then, this implies, as df_x is a linear isomorphism, that $df_x(T_x(X)) + df_x(T_x(Z)) = df_x(T_x(Y))$. As f is a diffeomorphism, we know that $df_x(T_x(Y)) = T_{f(x)}Y'$, $df_x(T_x(X)) = T_{f(x)}(f(X))$, and $df_x(T_x(Z)) = T_{f(x)}(f(Z))$. This shows that $f(X) \pitchfork f(Z)$. $\square$

We now apply this to curves in S^2.

Proposition 23.5. *Suppose that X, Z are submanifolds of S^2, and $X \cong Z \cong S^1$. Then, $I_2(X, Z) = 0$.*

Proof. We have previously shown that if we take $i_X : X \to S^2$ to be the inclusion map, then $y \in S^2$ is a regular value if and only if y is not in the image of i_X, since $\dim X < \dim S^2$. Similarly, y is a regular value of $i_Z : Z \to S^2$ if and only if y is not in the image of i_Z.

By Sard's Theorem, then there exists a $y \in S^2$ such that y is not in the image of either inclusion map. Hence, $y \notin X$ and $y \notin Z$. We now think of y as the north pole in a stereographic projection π_y of $S^2 \setminus \{y\}$ onto a plane $P \cong \mathbb{R}^2$.

We have that π_y is a diffeomorphism, so X and Z are mapped to diffeomorphic manifolds X' and Z' of $\mathbb{R}^2$. Now we are left with the previous situation of two submanifolds of $\mathbb{R}^2$, each diffeomorphic to S^1, so by the previous lemma, $I_2(X', Z') = 0$.

As π_y is a diffeomorphism, then $I_2(X, Z) = I_2(X', Z') = 0$. $\square$

We use this fact to show that S^2 and $T^2 = S^1 \times S^1$ are not diffeomorphic.

Theorem 23.6. *S^2 and T^2 are not diffeomorphic.*

Proof. We will find two submanifolds X, Z of T^2, each diffeomorphic to S^1, whose mod 2 intersection number is 1. This will show that T^2 is not diffeomorphic to S^2 – if it were, then the previous proposition says that the mod 2 intersection number $I_2(X, Z) = 0$, a contradiction.

But this is easy to do: Fix a point $p \in S^1$ and take $X = S^1 \times \{p\}$ and $Z = \{p\} \times S^1$. One can easily check that this intersection is transversal, and there is exactly 1 point in the intersection, namely (p, p). $\square$

A similar technique can be used to show that S^n and T^n are not diffeomorphic.

Theorem 23.7. *S^n and T^n are not diffeomorphic.*

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Theorem 24.1 (Borsuk-Ulam Theorem in dimension 1)**.** *Let $f : S^1 \to S^1$ be a smooth map such that $f(-x) = -f(x)$. Then,* $\deg_2(f) = 1$.

Proof. Let $y \in S^1$ be a regular value of $f(x)$ – we know such a y exists by Sard's Theorem. Removing y and $-y$ from S^1 makes the set $S^1 \setminus \{y, -y\}$ into a union of two components, U_1 and U_2.

Suppose that $\deg_2(f) = 0$. Then $f^{-1}(y)$ and $f^{-1}(-y)$ both have an even number of points, and $f^{-1}(\{y, -y\})$ consists of $4k$ points, for some $k \in \mathbb{N}$. Then, $S^1 \setminus f^{-1}(\{y, -y\})$ is a union of $4k$ components $V_1, \ldots, V_{4k}$ (here, we order the components counter-clockwise along the circle).

We first claim that for each i, $f(V_i)$ lies entirely within U_1 or entirely within U_2. Since V_i is connected, $f(V_i)$ is connected as well. But if $f(V_i)$ contains two points $z_1 \in U_1$ and $z_2 \in U_2$, then $f(V_i)$ must contain a path between z_1 and z_2. But such a path must pass through either y or $-y$.

We now also claim that the image of any two adjacent open sets V_i and V_{i+1} must lie in different U_j. Let z be the common endpoint of V_i and V_{i+1}. Since y is a regular value of $f(x)$, then $-y$ is also a regular value of $f(x)$, since the preimage of $-y$ is exactly $-f^{-1}(\{y\})$, and $df_{-x} = -df_x$ since $f(-x) = -f(x)$. Then, the fact that y and $-y$ are regular values of f, and $z \in f^{-1}(\{y, -y\})$ implies that $df_z : T_z(S^1) \to T_{\pm y}(S^1)$ is an isomorphism.

Since $T_z(S^1)$ is 1-dimensional, it is spanned by a single vector v. By Chapter 1 Section 2 Exercise 12, then v is the velocity vector of some curve $\gamma : [-\epsilon, \epsilon] \to S^1$, and $df_z(v) = f(\gamma(t))'|_{t=0}$. But if both V_i and V_{i+1} map to the same U_j, then $\gamma(t) \in U_j$ for all sufficiently small t. By continuity, we can find a sequence of $t_i < 0$ and $s_i > 0$ such that $t_i, s_i \to 0$ and $f(\gamma(t_i)) = f(\gamma(s_i))$. But then, this implies that $f(\gamma(t))'|_{t=0} = 0$, which contradicts that df_z is an isomorphism.

From the second claim, we see that $f(V_1)$ and $f(V_{2k+1})$ must both lie in the same U_j. WLOG, say that $j = 1$. On the other hand, the fact that $f(-x) = -f(x)$ implies that the set of endpoints of the V_i must be symmetric about the origin, so $V_{2k+1} = -V_1$. Therefore, $f(V_{2k+1}) = f(-V_1) = -f(V_1)$, which means that $f(V_{2k+1})$ must lie in U_2. $\square$

25 November 21, 2014

Theorem 25.1 (Borsuk-Ulam Theorem). *Let $f : S^k \to \mathbb{R}^{k+1}$ be a smooth map whose image does not contain the origin, and suppose that f satisfies the symmetry condition $f(-x) = -f(x)$ for all $x \in \mathbb{S}^k$. Then, $W_2(f, 0) = 1$.*

Proof. For $k = 1$, this follows from the one-dimensional case: if $f : S^1 \to \mathbb{R}^2$, then we can define $\tilde{f} : S^1 \to S^1$ by $\tilde{f}(x) = f(x)/|f(x)|$. Then $\tilde{f}$ still satisfies the symmetry condition, and

$$W_2(f, 0) = \deg_2\left(\frac{f(x) - 0}{|f(x) - 0|}\right) = \deg_2(\tilde{f}) = 1.$$

We proceed by induction. Suppose the theorem holds for $k - 1$. Suppose $f : S^k \to \mathbb{R}^{k=1} \setminus \{0\}$ satisfy the hypotheses of the theorem. Let S^{k-1} denote the equator $(x_1, \dots, x_k, 0)$. Let g be the restriction $f|_{S^{k-1}}$ of f to S^{k-1}. By Sard's Theorem, there exists a vector $\vec{a}$ that is a regular value for both $g/|g|$ and $f/|f|$. By the symmetry argument, we see that $-\vec{a}$ is also a regular value of both maps.

Since $g/|g|$ maps from a $k - 1$ dimensional manifold to a k dimensional manifold, regularity implies that neither $\vec{a}$ nor $-\vec{a}$ are in the image of $g/|g|$.

Furthermore, since $\vec{a}$ and $-\vec{a}$ are regular values of $f/|f|$,t hen $f/|f|$ is transversal to $\{\vec{a}, -\vec{a}\}$. Take $h(u) = u/|u|$. This means that $h \circ f \pitchfork \{\vec{a}, -\vec{a}\}$. By Chapter 1 Section 5 Exercise 7, then $f \pitchfork h^{-1}(\{\vec{a}, -\vec{a}\})$. But $h^{-1}(\{\vec{a}, -\vec{a}\}) = \operatorname{span}(\vec{a}) = l$.

By definition,

$$W_2(f, 0) = \deg_2(f/|f|) = \#(f/|f|)^{-1}(\vec{a}) \pmod 2.$$

By the symmetry of f, $\#(f/|f|)^{-1}(\vec{a}) = \#(f/|f|)^{-1}(-\vec{a})$, and each of these can be computed by the number of times f hits l in the upper hemisphere.

But the upper hemisphere is a manifold with boundary, with boundary $=$ S^{k-1}. Now, g is a map from S^{k-1} but the codomain has dimension too high to invoke the inductive hypothesis. Let V be the orthogonal complement of l and take $\pi : \mathbb{R}^{k+1} \to V$ be the orthogonal projection. As π is linear, it preserves the symmetry of g, so $\pi \circ g : S^{k-1} \to \mathbb{R}^k$ satisfies the symmetry condition. Moreover, $\pi \circ g$ is never equal to 0, since by assumption, $g/|g|$ does not have $\{\vec{a}, -\vec{a}\}$ in it image, and hence, g does not contain any point of $l = \ker \pi$ in its image.

The inductive hypothesis then implies that $W_2(\pi \circ g, 0) = 1$.

Recall the theorem from the proof of the Jordan Brouwer Separation Theorem:

Theorem 25.2. *Suppose that X is a $n-1$ dimensional manifold that is the boundary of a compact manifold with boundary, D, and let $F : D \to \mathbb{R}^n$ be a smooth map extending $f : X \to \mathbb{R}^n$, i.e. $\partial F = f$. Then $F^{-1}(z)$ is a finite set, and $W_2(f, z) = \#F^{-1}(z)$ mod 2. In other words, f winds X around z as often as F hits z, mod 2.*

This implies that $W_2(\pi \circ g, 0)$ is equal to the number of times $\pi \circ f$ hits 0, which is also equal to the number of times that f hits $\pi^{-1}(0) = l$ in the upper half sphere. We previously argued this is the same as $W_2(f, 0)$. □

We obtain the following applications:

Theorem 25.3. *If $f : S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ is symmetric about the origin, i.e. $f(-x) = -f(x)$, then f intersects every line through the origin at least once.*

Proof. If f never hits l, then f is transversal to l, but this contradicts the part of the above proof that f hits l, 1 mod 2 times. □

Theorem 25.4. *Any k smooth functions $f_1, \ldots, f_k$ on $\mathbb{S}^k$ that satisfy the symmetric condition $f_i(-x) = -f_i(x)$ must possess a common zero.*

Proof. Apply the previous theorem to $f(x) = (f_1(x), \ldots, f_k(x), 0)$, with l being the x_{k+1} axis. But the x_{k+1} axis is defined to be the points where $x_1 = x_2 = \cdots = x_k = 0$, so if f hits l,t hen $f_1(x) = \cdots = f_k(x) = 0$. □

Theorem 25.5. *For any k smooth functions $g_1, \ldots, g_k$ on S^k there exists a point $p \in S^k$ such that $g_1 = g_1(-p), \ldots, g_k(p) = g_k(-p)$.*

Proof. Take $f_i(x) = g_i(x) - g_i(-x)$. Then f_i satisfies the symmetry condition, so by the previous theorem, there is a common zero. This implies that $0 = g_i(x) - g_i(-x)$, or $g_i(x) = g_i(-x)$. □

26 November 24, 2014

Theorem 26.1 (ϵ neighborhood Theorem)**.** *For a compact boundaryless manifold $Y \subset \mathbb{R}^M$ and $\epsilon > 0$, let $Y^\epsilon = \{w \in \mathbb{R}^M : d(y, w) < \epsilon \text{ for some } y \in Y\}$ be the ϵ neighborhood of Y. If ϵ is sufficiently small, then each point $w \in Y^\epsilon$ has a unique closest point in Y, denoted $\pi(w)$, and $\pi : Y^\epsilon \to Y$ is a submersion. If Y is not compact, then there still exists a submersion $\pi : Y^\epsilon \to Y$ that is the identity on Y, but ϵ must be allowed to vary as a smooth function on Y, and $Y^\epsilon = \{w \in \mathbb{R}^M : d(w, y) < \epsilon(y) \text{ for some } y \in Y\}$.*

We will introduce the normal space and the normal bundle, which are like the tangent space and tangent bundle.

Definition 26.2. Let $X \subset \mathbb{R}^M$ be a smooth manifold and $x \in X$. Then, the *normal space* of X at x, $N_x(X)$ is the orthogonal complement to $T_x(X)$ in $\mathbb{R}^M$.

The normal bundle $N(X)$ is the set

$$N(X) = \{(x, v) : v \in N_x(X)\}.$$

The tangent space and tangent bundle are intrinsic to X – for different embeddings into $\mathbb{R}^M$ for different M, we get isomorphic/diffeomorphic spaces. But for the normal space and normal bundle, they depend very much on the ambient space. If the dimension of the ambient space is larger, then the dimension of the normal space/bundle will be larger.

Proposition 26.3. *If $X \subset \mathbb{R}^M$, then $N(X)$ is a manifold of dimension M, and $\sigma : N(Y) \to Y$ is a submersion.*

Lemma 26.4. *Let $A : \mathbb{R}^M \to \mathbb{R}^k$ is a linear map (i.e. a matrix). If A is surjective, then A^T maps $\mathbb{R}^k$ isomorphically onto the orthogonal complement of* $\ker A$.

Proof. We first show A^T is injective. Suppose $A^T w = 0$. Then, for every $v \in \mathbb{R}^M$, we have that $Av \cdot w = v \cdot A^T w = v \cdot 0 = 0$. But A is surjective, so this must mean that $w = 0$.

Now we show that A^T is orthogonal to $\ker A$. Suppose $Av = 0$. Then, for every $w \in \mathbb{R}^k$, $Av \cdot w = v \cdot A^T w = 0$. Hence, $A^T w$ is orthogonal to the kernel of A, and the image of A^T is contained in the orthogonal complement of $\ker A$. But as A is surjective, $\ker A$ has dimension $M - k$, and its orthogonal complement has dimension k. So the image of A^T must be the entire orthogonal complement. □

Proof of Proposition. Let $x \in X$ and take $\tilde{U}$ to be an open subset of $\mathbb{R}^M$ that contains x. We know that X can be written as the zero set of some function, so take $\phi : \tilde{U} \to \mathbb{R}^{\operatorname{codim} X}$ so that $U = X \cap \tilde{U} = \phi^{-1}(0)$. We have that $N(U) = N(X) \cap (U \times \mathbb{R}^M)$ is open in $N(X)$.

Then $\psi : U \times \mathbb{R}^{\operatorname{codim} X} \to N(U)$ defined by $\psi(x, v) = (x, d\phi_x^T v)$ gives the desired parametrization of $N(U)$.

σ is the submersion because $\sigma \circ \psi$ is the standard submersion. □

Epsilon neighborhood theorem. Let $h : N(Y) \to \mathbb{R}^M$ be defined by $h(y, v) = y + v$. Note that h is regular at every point in $Y \times \{0\}$. This is because $dh_{(y,0)}$ splits into two pieces, the first part acting on $T_y(Y)$, mapping it into $T_y(Y)$ (this is from the y part), and the second part acting on $N_y(Y)$, mapping it into $N_y(Y)$ (this is from the v part). Since $T_y(Y) + N_y(Y) = \mathbb{R}^M$, this is surjective.

Now h maps $Y \times \{0\}$ diffeomorphically onto Y and is regular at each $(y, 0)$, so it maps a neighborhood of $Y \times \{0\}$ diffeomorphically onto a neighborhood of Y in $\mathbb{R}^M$ (for Y compact, this is the generalized inverse function theorem from I.3.10 on Homework #4. The general case is Exercise I.8.14). Any neighborhood of Y contains some Y^ϵ (again, we have proved this in the compact case, the general case requires additional techniques). Then, $h^{-1} : Y^\epsilon \to N(Y)$ is defined, and $\pi = \sigma \circ h^{-1} : Y^\epsilon \to Y$ is the desired submersion. □

We now discuss the main tool from going from compact manifolds to non-compact manifolds: partitions of unity.

Theorem 26.5. *Let X be an arbitrary subset of $\mathbb{R}^N$. For any covering of X by subsets $\{U_\alpha\}$ open in X, there exists a sequence of smooth functions $\{\theta_i\}$ on X, called a partition of unity subordinate to the cover $\{U_\alpha\}$ with the following properties:*

1. $0 \le \theta_i(x) \le 1$ *for all* $x \in X$ *and all* i
2. *Each $x \in X$ has a neighborhood on which all but finitely many functions θ_i are identically zero.*
3. *Each function θ_i is identically zero except on some closed set contained in one of the U_α.*
4. *For each $x \in X$, $\sum_i \theta_i(x) = 1$.*

Proof. If U_α is open in X, then there exists W_α open in $\mathbb{R}^N$ such that $U_\alpha = X \cap W_\alpha$. Let $W = \cup_\alpha W_\alpha$. Take $\{K_j\}$ be a sequence of compact sets $K_j \subset \operatorname{Int} K_{j+1}$ such that $\cup_j K_j = W$. Such a sequence exists – for example, one can take $K_j = \{z \in W : |z| < j \text{ and } d(z, W^C) \geq 1/j\}$.

Take the set of open balls in $\mathbb{R}^N$ whose closure lies entirely inside at least one of the W_α. As W_α is open in $\mathbb{R}^N$, this forms a cover of W – if $x \in W$, then $x \in W_\alpha$ for some α. Then, take an ϵ such that $B_\epsilon(x) \subset W_\alpha$, and take the $\epsilon/2$ ball about x, whose closure is contained in W_α.

By the compactness of K_2, we can consider the cover of W as a cover of K_2 and take a finite subcover. By Chapter 1 Section 1 Exercise 18, we know that there exists a function that is identically 1 on a given ball and identically 0 on a given ball of larger radius, and between 0 and 1 in between. So for each ball in the finite sub cover, there exists a smooth nonnegative function on $\mathbb{R}^N$ that is equal to one on the ball and equal to 0 outside of a closed ball that is contained in W_α. We call these functions $\eta_1, \ldots, \eta_r$.

We now proceed inductively – for each $j \geq 3$ the subset $K_j \setminus \operatorname{Int} K_{j-1}$ is a compact set that is contained inside the open set $W \setminus K_{j-2}$. Take the set of all open balls whose closure is contained wholly in $W \setminus K_{j-2}$ and some W_α – this forms a cover of $K_j \setminus \operatorname{Int} K_{j-1}$. Then take a finite sub cover, and for each ball, add another function η_s for each ball that is 1 on the ball and 0 on a closed ball contained in both $W \setminus K_{j-2}$ and in some W_α.

For each j, only finitely many function n_s are nonzero on K_j. Hence, $\sum_s \eta_s$ is finite in a neighborhood of every point of W, and at least one term is nonzero at any point of W, as the sequence of finite subcovers cover W. Hence,

$$\frac{\eta_i}{\sum_s \eta_s}$$

is well-defined and smooth. Let θ_i be the restriction to X, and $\{\theta_i\}$ satisfies the desired properties. □

We obtain the important corollary:

Corollary 26.6. *On any manifold X, there exists a proper map $p : X \to \mathbb{R}$.*

Proof. Let $\{U_\alpha\}$ be the collection of open subsets of X whose closure is compact. Let θ_i be a partition of unity subordinate to the cover $\{U_\alpha\}$. Then,

$$p = \sum_{i=1}^{\infty} i\theta_i$$

is a well-defined smooth function since only finitely many θ_i are nonzero at a given point.

If $p(x) \leq j$, then it must be that for all $i > j$, that $\theta_i(x) = 0$. Since at each point x, $\sum_i \theta_i(x) = 1$, this means that at least one of the first j functions $\theta_1, \dots, \theta_j$ is nonzero at x.

Then $p^{-1}([-j, j])$ is contained in

$$\cup_{i=1}^{j} \{x : \theta_i(x) \neq 0\}.$$

The above set has compact closure, as it is contained in the union of at most finitely many of the U_α. Every compact subset of $\mathbb{R}$ is contained in some $[-j, j]$, so this shows that the inverse image of every compact subset is compact. $\square$

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As an example of an application of partitions of unity, we will apply them to extend the generalized inverse function to noncompact manifolds. Recall the compact version:

Proposition 27.1 (Chapter 1 Section 3 Exercise 10)**.** *Let $f : X \to Y$ be a smooth map that is one-to-one on a compact submanifold Z of X. Suppose that for all $x \in Z$, $df_x : T_x(X) \to T_{f(x)}(Y)$ is an isomorphism. Then f maps an open neighborhood of Z in X diffeomorphically onto an open neighborhood of $f(Z)$ in Y.*

We wish to prove the noncompact version (cf Chapter 1 Section 8 Exercise 14):

Proposition 27.2 (Chapter 1 Section 8 Exercise 14)**.** *Suppose $f : X \to Y$ maps a submanifold Z diffeomorphically onto $f(Z)$ and that for each $x \in Z$, df_x is an isomorphism. Then f maps a neighborhood of Z diffeomorphically onto a neighborhood of $f(Z)$.*

We begin with a definition and a lemma:

Definition 27.3. An open cover $\{U_\beta\}$ of X is a *refinement* of the open cover $\{V_\alpha\}$ if for every α, there exists a β such that $U_\alpha \subset V_\beta$.

Definition 27.4. An open cover $\{U_\alpha\}$ of X is *locally finite* if every $x \in X$ has a neighborhood that intersects only finitely many of the U_α.

Lemma 27.5 (Chapter 1 Section 8 Exercise 13)**.** *Every open cover* $\{V_\alpha\}$ *of a manifold* X *has a locally finite refinement.*

Proof. Let $\{\theta_i\}$ be a partition of unity subordinate to V_α. Take $U_i = \theta_i^{-1}((0,\infty))$. As θ_i is smooth, U_i is open. Since $\{U_i\}$ is a partition of unity, for each $x \in X$, there exists an i such that $\theta_i(x) > 0$, so $x \in U_i$ for some i. Hence, $\{U_i\}$ is a cover of X.

Moreover, as $\{\theta_i\}$ is subordinate to V_α, each θ_i has is identically zero except on a closed set contained wholly inside one of the V_α, so $U_i \subset V_\alpha$ for some α. In other words, $\{U_i\}$ is a refinement of the cover $\{V_\alpha\}$.

Finally, each $x \in X$ has a neighborhood on which all but finitely many of the θ_i are identically zero, so this neighborhood only intersects finitely many of the U_i. $\square$

We now prove the proposition.

Chapter 1 Section 8 Exercise 14. Suppose $f : X \to Y$ maps $Z \subset X$ diffeomorphically onto $f(Z) \subset Y$.

Since $df_x : T_x(X) \to T_y(y)$ for each $x \in Z$, there exist open sets $x \in U_x \subset X$ and $V_x \subset Y$ such that f maps U_x diffeomorphically onto V_x, by the inverse function theorem for manifolds.

The $\{V_x\}_{x\in X}$ forms an open cover of $f(Z)$, since $f(x) \in V_x$. Now, apply the lemma to take a locally finite refinement, which we will call V_i, with local inverse $g_i : V_i \to X$, such that $f \circ g_i = Id_{V_i}$.

Define $W = \{y \in Y : g_i(y) = g_j(y) \text{ whenever } y \in V_i \cap V_j\}$. On W, it is clear we can define a global inverse $g : W \to X$ by taking $g(y) = g_i(y)$ for any i such that $y \in V_i$. This is well defined on W as $g_i(y) = g_j(y)$ whenever $y \in V_i \cap V_j$.

W contains Z, as f maps Z diffeomorphically on $f(Z)$, so $g_i(y) = g_j(y) = f^{-1}(y)$ for any $y \in Z$. Now fix a $f(x) \in f(Z)$. We wish to show that W contains an open neighborhood of $f(x)$. By the property that $\{V_i\}$ is a locally finite cover of $f(Z)$, there exists a neighborhood V of $f(x)$ that intersects only finitely many of the V_i – by reindexing them, if necessary, call them $V_1, \ldots, V_k$.

Then $\tilde{V} = V \cap V_1 \cap \cdots \cap V_k$ is a finite intersection of open sets that contain $f(x)$, so $\tilde{V}$ is an open neighborhood of $f(x)$. Moreover, on $\tilde{V}$ each g_i is a local diffeomorphism with $g(\tilde{V}) \subset U_i \ni x$. Hence, on $\tilde{V}$, the g_i all agree, so $\tilde{V} \subset W$. This proves the proposition. $\square$

We will now discuss how partitions of unity can be used to prove the Whitney embedding theorem for noncompact manifolds. Recall that we proved:

Corollary 27.6. *On any manifold X, there exists a proper map $p : X \to \mathbb{R}$.*

Recall that the Whitney immersion theorem, which says that X^k has an injective immersion into $\mathbb{R}^k$ was proved inductively by projecting along a vector to get an injective immersion into a lower dimensional space. If X is compact, an injective immersion is automatically proper, so it is an embedding.

Theorem 27.7 (Weak Whitney Embedding Theorem)**.** *Every k-dimensional manifold embeds into $\mathbb{R}^{2k+1}$.*

Proof. We first use the Whitney immersion theorem, which says that for X^k, there exists an immersion $f : X^k \to \mathbb{R}^{2k+1}$. Compose with the diffeomorphism

$$g : \mathbb{R}^{2k+1} \to B_1(0) : z \mapsto \frac{z}{1 + |z|^2}.$$

This gives an injective immersion $g \circ f : X \to \mathbb{R}^{2k+1}$ such that $|g \circ f(x)| < 1$ for all $x \in X$. Take $p : X \to \mathbb{R}$ to be the proper function from the corollary. We define an injective immersion $F(x) = (g \circ f(x), p(x))$.

We then do the projection $\pi : \mathbb{R}^{2k+2} \to H$ along a vector a onto its orthogonal complement H, as in the Whitney immersion theorem, so that $\pi \circ F$ is still an injective immersion.

The claim is that the properness of p actually now makes $\pi \circ F$ proper. This is proved by showing that if $|\pi \circ F(x)| \leq c$,t hen there exists a d such that $|p(x)| \leq d$. Since p is proper, this will show properness of $\pi \circ F$. The details are in Guillemin and Pollack. $\square$