Proof of Jordan-Brouwer Separation Theorem

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1. **Show that if F does not hit z, then $W_2(f, z) = 0$**

   Suppose $z \in \mathbb{R}^n - F(D)$. Then, we can define the unit vector mapping u on X by
   $$u(x) = \frac{f(x) - z}{|f(x) - z|}.$$
   As stated, this formula is valid for both X and D, since we assumed that $z \notin F(D)$; therefore, we can define $U : D \to Y$ with $U|_X = u$, by using the formula
   $$U(x) = \frac{F(x) - z}{|F(x) - z|}.$$
   Furthermore, U is smooth because it is the composition of $x \mapsto F(x) - z$ and $x \mapsto x/|x|$, which are both smooth since the latter is defined on $F(D) - z$, which doesn't include 0. This proves that U is an extension of u to all of D, so by the theorem on p. 81 in the book, we have $\deg_2(u) = 0$. Thus,
   $$W_2(f, z) = \deg_2(u) = 0.$$

2. **Suppose that $F^{-1}(z) = \{y_1, \ldots, y_l\}$, and around each point y_i let B_i be a ball (that is B_i is the image of a ball in $\mathbb{R}^n$ via some local parametrization of D). Demand that the balls be disjoint from one another and from $X = \partial D$. Let $f_i : \partial D_i \to \mathbb{R}^n$ be the restriction of F, and prove that**

   $$W_2(f, z) = W_2(f_1, z) + \cdots + W_2(f_l, z) \textbf{ mod 2}.$$

The values $y_1, \ldots, y_l$ are all distinct, and the condition $z \notin f(X)$ implies that $y_1, \ldots, y_l \notin X$, so we can demand that the balls are disjoint and separated from one another and from X, since $\mathbb{R}^n$ is a regular (T3) Hausdorff space. Define

$$D' = D - \bigcup_{i=1}^{l} \mathrm{Int}(B_i).$$

Note that by shrinking each B_i, we can require that they are contained in $\mathrm{Int}(D)$, so each ∂B_i (here we mean the topological boundary) is contained in $\mathrm{Int}(D)$ as well. ∂B_i is a $n-1$ dimensional manifold, so by the local immersion theorem, there exists a local parametrization $\phi : \mathbb{R}^n \to V$ where $\phi(0) = y \in V$, and ∂B_i, locally, is the subset $\mathbb{R}^{n-1}$ of $\mathbb{R}^n$ where the nth coordinate is 0. Therefore, $\partial B_i \subseteq \partial D'$ (here, we mean the manifold boundary). Since X is separated from the balls, we can shrink any local parametrization of a point $y \in \partial D \subseteq D$ so that it is also a local parametrization of $y \in D'$ with y on the boundary. This shows that $\partial D \subseteq \partial D'$. Since all other points in D' are interior points, we've proven that

$$\partial D' = \partial D \cup \bigcup_{i=1}^{l} \partial B_i.$$

Let $F_{D'}$ be the restriction of F to D'. By contruction, $F_{D'}$ never hits z, so letting $f_{\partial D'} = \partial F_{D'}$, we have by Exercise 1 that $0 = W_2(f_{\partial D'}, z)$. But if $u : \partial D' \to S^{n-1}$ is the unit vector map induced by $f_{\partial D'}$ and z, then for any $y \in S^{n-1}$,

$$0 = W_2(f_{\partial D'}, z) = \deg_2(u) = I_2(u, \{y\}) = \#[u^{-1}(y)] \mod 2. \quad (1)$$

Since $X = \partial D, \partial B_1, \ldots, \partial B_l$ are all disjoint and have union $\partial D'$, we have

$$0 = \#[u^{-1}(y)] = \#[u^{-1}(y) \cap \partial D] + \#[u^{-1}(y) \cap \partial B_1] + \cdots + \#[u^{-1}(y) \cap \partial B_l].$$

Clearly $u^{-1}(y) \cap Z = {u_Z}^{-1}(y)$, where u_Z is the unit vector map induced by z and the restriction of $f_{\partial D'}$ to the submanifold $Z \subseteq \partial D'$, so the above translates to

$$0 = \#[{u_{\partial D}}^{-1}(y)] + \#[{u_{\partial B_1}}^{-1}(y)] + \cdots + \#[{u_{\partial B_l}}^{-1}(y)],$$

which translates by (1) to

$$0 = W_2(f,z) + W_2(f_1,z) + \cdots + W_2(f_l,z) \mod 2.$$

Rearranging, and noting that $a = -a \mod 2$, we conclude that

$$W_2(f,z) = W_2(f_1,z) + \cdots + W_2(f_l,z) \mod 2.$$

3. **Use the regularity of z to choose the balls B_i so that $W_2(f_i,z) = 1$, and thus prove the theorem.**

   Since z is a regular value of F, we have that $dF_z : D \to Y$ is surjective. It was presumed that $\dim D = n = \dim Y$, so dF_z is thus bijective, and thus an isomorphism, so by the inverse function theorem, F is a local diffeomorphism at each y_i.

   So for each i, we can find an open subset $U \subseteq D$ with $y_i \in U$ and open $V \subseteq \mathbb{R}^n$ with $z \in V$ such that $F : U \to V$ is a diffeomorphism. By openness, we can pick a closed ball $A_i \subseteq V$ centered at z such that the restriction of F to some closed set $B_i \subseteq U$ (with $y_i \in B_i$) maps diffeomorphically to A_i. Exercise 2.1.2 tells us that the restriction of F to ∂B_i (call it f_i) maps ∂B_i diffeomorphically to ∂A_i.

   ∂A_i is a small sphere centered at z, so for any $w_1, w_2 \in \partial B_i$, we have $f_i(w_1), f_i(w_2) \in \partial A_i$, and thus we may write them uniquely as

$$f_i(w_1) = z + rv_1 \& f_i(w_2) = z + rv_2$$

   where r is the radius of A_i and $v_1, v_2 \in S^{n-1}$. But then we have we have, with $u : \partial B_i \to S^{n-1}$ denoting the unit vector map induced by f_i and z, and if $u(w_1) = u(w_2)$, then

$$\begin{aligned} u(w_1) &= u(w_2) \\ \Rightarrow \frac{f_i(w_1) - z}{|f_i(w_1) - z|} &= \frac{f_i(w_2) - z}{|f_i(w_2) - z|} \\ \Rightarrow \frac{rv_1}{|rv_1|} &= \frac{rv_2}{|rv_2|} \\ \Rightarrow v_1 &= v_2 \\ \Rightarrow z + rv_1 &= z + rv_2 \\ \Rightarrow w_1 &= w_2, \end{aligned}$$

so u is injective. It's also surjective because as we showed just above,

$$[v \in S^{n-1}] \implies [z + rv \in \partial A_i] \implies [u(f_i^{-1}(z + rv)) = v \in S^{n-1}].$$

Since u is bijective, we have for any $v \in S^{n-1}$ that $u^{-1}(v)$ contains a single point. But then

$$1 = \#[u^{-1}(v)] = I_2(u, \{v\}) = \deg_2(u) = W_2(f_i, z),$$

for each i. Therefore,

$$W_2(f, z) = W_2(f_1, z) + \cdots + W_2(f_l, z) = l = \#[F^{-1}(z)] \mod 2.$$

4. **Let $z \in \mathbb{R}^n \setminus X$. Prove that if x is any point of X and U any neighborhood of x in $\mathbb{R}^n$, then there exists a point of U that may be joined to z by a curve not intersecting X.**

   Fix $z \in \mathbf{R}^n - X$. Let S be the set of all points of X such that the above is true. First we will show that S is closed in X.

   Let (s_i) be a sequence of points of S which converge to some point s in X, $s_i \to s$. If we have that $s \in S$, then we are done. Assume that $s \notin S$.

   Let U be an open neighborhood of s in $\mathbf{R}^n$, $s \in U$. Clearly we have some $s_i \in U$. But then there exists some open neighborhood U' of s_i, $s_i \in U'$. Without loss of generality, we may assume that U' is small enough such that $U' \subset U$. Then, because $s_i \in S$, there must exist a point of U' that may be joined to z by a curve not intersecting X. But this this implies such a point exists in U, which is a contradiction, as we have assumed that $s \notin S$. Thus, we must have that $s \in S$, and hence S is closed.

   Now we may show that S is open. We will do so by showing that every point in S is an interior point. Let $s \in S$ be any point. From the inclusion map $i : X \to \mathbf{R}^n$, the Local Immersion Theorem tells us that there exists local coordinates on a neighborhood U of s in $\mathbf{R}^n$ such that $U \cap X$ is equal to the set of all points of the form $(u_1, \ldots, u_{n-1}, 0)$.

   Now, because $s \in S$, there exists a point $\tilde{u} \in (U - X)$ such that $\tilde{u}$ can be connected to z by a curve. Write $\tilde{u} = (u_1, \ldots, u_n)$, with $u_n \neq 0$, as $\tilde{u} \notin X$. Thus, u_n may either be positive or negative. Without loss of generality, let us assume that $u_n > 0$. Now, take any point $s_1 \neq s$, with $s_1 \in U \cap X$, and take any neighborhood V of s_1 in $\mathbf{R}^n$. We have that $V \cap U \neq \emptyset$. Thus, there exists some point $v \in V \cap U$ such that the nth coordinate of v is positive. Because $v \in U$, we can join v to $\tilde{u}$ by a curve that doesn't intersect X. We can then concatenate this curve from v to $\tilde{u}$ with the curve from $\tilde{u}$ to z, deforming slightly at $\tilde{u}$ if needed to maintain smoothness. This new curve from v to z does not intersect X, and thus $s_1 \in S$.

This shows us that $(U \cap X) \subset S$. Because $U \cap X$ is open in X, we have that s is an interior point. Thus, S must be open, as every element of s is an interior point.

5. **Show that $\mathbb{R}^n \setminus X$ has, at most, two connected components.**

   Again, using the local immersion theorem at a point $x \in X$, there is a neighborhood U of x where the inclusion of X into $\mathbb{R}^n$ is locally equivalent to the the canonical immersion $(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, 0)$. Without loss of generality, we let x map to 0. Then, we can take a small ball $B_\epsilon(0)$. Now, $B_\epsilon(0) \setminus X$ has two components, the upper half sphere and lower half sphere. Take z_0, z_1 in each component. WLOG, we also shrink U so that it is diffeomorphic to B_ϵ.

   Now let $y \in \mathbb{R}^n \setminus X$ be an arbitrary point. By the previous exercise, there exists a point in U that is connected to y through a path not intersecting X. But each point in U is connected to either z_0 or z_1 by a path not intersecting X, so we can concatenate the paths to get a path from y to either z_0 or z_1.

   Hence, $\mathbb{R}^n \setminus X$ has at most 2 components, one containing z_0 and the other z_1.

6. **Show that if z_0 and z_1 belong to the same connected component of $\mathbb{R}^n \setminus X$, then $W_2(X, z_0) = W_2(X, z_1)$.**

   Since z_0 and z_1 belong to the same connected component which is a subset of $\mathbb{R}^n$-X, there exists a path between z_0 and z_1,

   $$z_t : [0, 1] \mapsto U \tag{2}$$

   with $z_t(0)=z_0$ and $z_t(1)=z_1$
   Consider such a curve z_t in $\mathbb{R}^n$-X connecting z_0 to z_1. We have the

following map which is clearly a homotopy:

$$u_t(x) = \frac{x - z_t}{|x - z_t|} \tag{3}$$

It is always defined since $z_t \in \mathbb{R}^n$-X, and thus $z_t \neq x$.
u_0 and u_1 are homotopic and homotopic maps have same mod 2 degree. This implies that $\deg_2(u_0) = \deg_2(u_1)$ and consequently, $W_2(x, z_0) = W_2(x, z_1)$.

7. **Given a point $z \in \mathbb{R}^n \setminus X$ and a direction vector $v \in S^{n-1}$, consider the ray r emanating from z in the direction of v,**

$$r = \{z + tv : t \geq 0\}.$$

Check that the ray r is transversal to X if and only if v is a regular value of the direction map $u : X \to S^{n-1}$. In particular, almost every ray from z intersects X transversally.

The following exercise, Chapter 1 Section 5 Exercise 7, was proven by a student during the review for the first midterm :

Let $X \xrightarrow{f} Y \xrightarrow{h} Z$ be a sequence of smooth maps of manifolds, and assume that h is transversal to a submanifold W of Z. $f \pitchfork h^{-1}(W)$ if and only if $h \circ f \pitchfork W$.

Consider a point $z \in \mathbb{R}^n - X$ and a direction vector $\vec{v} \in S^{n-1}$. Define r to be the set $r := \{z + t\vec{v} : t \geq 0\}$. Also, let $g : \mathbb{R}^n - \{z\} \mapsto S^{n-1}$ be defined by $g(y) = \frac{y-z}{|y-z|}$. Note that $u : X \mapsto S^{n-1}$ is simply g composed with the inclusion map $i : X \mapsto \mathbb{R}^n$.

First, note that $g^{-1}(\{v\}) = r$. This is because if $x \in r$, then $x = z + t\vec{v}$ for some $t \geq 0$. Hence, $g(x) = \frac{z+t\vec{v}-z}{|z+t\vec{v}-z|} = \frac{t\vec{v}}{|t\vec{v}|}$ and since $\vec{v} = 1$, then $g(x) = \vec{v}$ and consequently $x \in g^{-1}(\{\vec{v}\})$. Thus, $r \subset g^{-1}(\{v\})$. For the other direction, suppose that $x \in g^{-1}(\vec{v})$. Thus, $\vec{v} = \frac{x-z}{|x-z|}$. Letting $t = |x - z|$ (where t some real number greater than or equal to 0), then $\vec{v} = \frac{x-z}{t}$. Rearranging, we see that $x = z + t\vec{v}$. Thus, $g^{-1}(\{v\}) \subset r$.

To apply the previously proved problem, we first need to see that $\vec{v}$ is a regular value of g. Since g goes from $\mathbb{R}^n - \{y\}$ to S^{n-1}, the derivative map dg_x goes from $T_x(\mathbb{R}^n - \{y\})$ to $T_{\vec{v}}(S^{n-1})$. The tangent space of $\mathbb{R}^n - \{y\}$ at x is n-dimensional, and the tangent space of S^{n-1} is $(n-1)$-dimensional.

If $\vec{v}$ is the direction from x to y, then $\vec{v}$ is the only direction in which g remains constant at x. Thus $\ker(dg_x) = \operatorname{span}(\vec{v})$, and so the dimension of the nullspace of dg_x is 1. By the rank-nullity theorem, the rank of dg_x is $n-1$, which equals the dimension of the codomain. Thus g is a submersion at any point $x \in g^{-1}(\vec{v})$.

Now, consider the sequence of smooth maps of manifolds $X \xrightarrow{i} \mathbb{R}^n \xrightarrow{g} S^{n-1}$. As we have shown, g is transversal to the submanifold $\{\vec{v}\}$ of S^{n-1}. Thus, $i \pitchfork g^{-1}(\{\vec{v}\}) \iff g \circ i \pitchfork \{\vec{v}\}$. Since $g^{-1}(\vec{v}) = r$, then $i \pitchfork g^{-1}(\{\vec{v}\}) \iff i \pitchfork r$. By the definition of sets being transversal, $i \pitchfork r \iff X \pitchfork r$. By construction $g \circ i = u$. Thus, $g \circ i \pitchfork \{\vec{v}\} \iff u \pitchfork \{\vec{v}\}$. Combining all of these equivalences, $u \pitchfork \{\vec{v}\} \iff X \pitchfork r$.

By Sard's Theorem almost every element $\vec{v}$ of the codomain of u is a regular value of u. From the result of the previous paragraph almost every ray from z intersects X tranversally.

8. **Suppose that r is a ray emanating from z_0 that intersects X transversally in a nonempty (necessarily finite) set. Suppose that z_1 is any other point on r (but no on X), and l be the number of times r intersects X between z_0 and z_1. Verify that $W_2(X, z_0) = W_2(X, z_1) + l \bmod 2$.**

   Let $u_0 = (f(x) - z_0)/|(f(x) - z_0)|$ and $u_1 = (f(x) - z_1)/|(f(x) - z_1)|$. By definition, $W_2(X, z_0) = \#u_0^{-1}(v) \bmod 2$, and $W_2(X, z_1) = \#u_1^{-1}(v) \bmod 2$.

   By exercise 7, v is a regular value for u_0 and u_1. This implies that $\#u_0^{-1}$ is finite, as $dimX = dim(S^{n-1})$, as is $\#u_1^{-1}$. X is compact.

   $\#u_0^{-1}$ equals the number of intersections of u_0 with X along v from z_0. It also equals the number of intersections from z_0 to z_1 plus the number of intersections of u_0 with X along v starting from z. This in turn equals l plus the number of intersections of u_1 and X along v, which equals $l + \#u_1^{-1}(v)$.

   Thus in conclusion, $W_2(X, z_0) = (\#u_1^{-1}(v) + l) \bmod 2$, which equals $(W_2(X, z_1) + l) \bmod 2$, as desired.

9. **Conclude that $\mathbb{R}^n \setminus X$ has precisely two components, $D_0 = \{z : W_2(X, z) = 0\}$ and $D_1 = \{z : W_2(X, z) = 1\}$.**

   We first establish that there exist $z_0, z_1 \in \mathbb{R}^n \setminus X$ such that $W_2(X, z_0) \neq W_2(X, z_1)$. Let $z_0 \in \mathbb{R}^n \setminus X$ be arbitrary and let r be a ray emanating from z_0 intersecting X transversally; such a ray exists by Exercise #7. Let z_1 be a point on the ray such that there is exactly one point of X on r between z_0 and z_1. Specifically, $r = \{z_0 + t\vec{v} : t \in [0, \infty)\}$, where $\vec{v}$ is is the direction of r. Then, if there is exactly one point where r intersects X (say, at $t = t_0$) we may let $z_1 = z_0 + 2t_0\vec{v}$. If there are at least two points of intersection (say, the first at $t = t_1$ and the second at $t = t_2$ where $t_1 < t_2$), we may let $z_1 = z_0 + \frac{t_1+t_2}{2}\vec{v}$.

By Exercise #8, then, $W_2(X, z_0) = W_2(X, z_1) + 1 \pmod 2$. Therefore, $W_2(X, z_0) \neq W_2(X, z_1)$. Suppose without loss of generality that $W_2(X, z_0) = 0$ and $W_2(X, z_1) = 1$. Now, by Exercise #6, z_0 and z_1 necessarily belong to two distinct connected components of $\mathbb{R}^n \setminus X$; let D_0 be the connected component with $z_0 \in D_0$ and D_1 that with $z_1 \in D_1$. Since by Exercise #5 $\mathbb{R}^n \setminus X$ has at most two connected components, we have that the connected components of $\mathbb{R}^n \setminus X$ are precisely D_0 and D_1. By Exercise #6, we have that $D_0 \subseteq \{z : W_2(X, z) = 0\}$ and $D_1 \subseteq \{z : W_2(X, z) = 1\}$.

We now consider the reverse inclusion. Note now that because $D_0 \cup D_1$ forms a partition of $\mathbb{R}^n \setminus X$, each $z \in \mathbb{R}^n \setminus X$ must belong to either D_0 or D_1. Suppose that $z \in \mathbb{R}^n \setminus X$ is such that $W_2(X, z) = 0$; by Exercise #6, z cannot be an element of D_1 as $W_2(X, z)$ and $W_2(X, z_1)$ differ; similarly, if $W_2(X, z) = 1$, z cannot be an element of D_0 as $W_2(X, z)$ and $W_2(X, z_0$ differ. Hence, we have that $D_0 \supseteq \{z : W_2(X, z) = 0\}$ and $D_1 \supseteq \{z : W_2(X, z) = 1\}$. Therefore, with both inclusions, we then have that $D_0 = \{z : W_2(X, z) = 0\}$ and $D_1 = \{z : W_2(X, z) = 1\}$.

10. **Show that if z is very large, then $W_2(X, z) = 0$.**

    By definition, $W_2(X, z) = \deg_2(u_z)$, where $u_z(x) = \frac{x-z}{|x-z|}$. Because S^{n-1} is connected, by the definition of degree, $W_2(X, z)$ is $\#\{x : \frac{x-z}{|x-z|} = y\} \pmod 2$, for any $y \in S^{n-1}$. Now, consider the map $u_z(x)$. We have that

    $$||u_z(x) - \frac{-z}{||z||}|| = \left\|\frac{x-z}{||x-z||} + \frac{z}{||z||}\right\| = \frac{1}{||z|| \cdot ||x-z||} \cdot \Big\| ||z||(x-z) + ||x-z||z \Big\|.$$

    Rewriting this expression and applying the triangle inequality gives that the quantity above is

    $$\frac{1}{||z|| \cdot ||x-z||} \cdot \Big\| ||z||x + z(||z-x|| - ||z||) \Big\| \leq \frac{1}{||z|| \cdot ||x-z||} \cdot \left(\Big\| ||z||x \Big\| + \Big\| z(||z-x|| - ||z||) \Big\| \right)$$

$$\leq \frac{2\|z\| \cdot \|x\|}{\|z\| \cdot \|x-z\|} = \frac{2\|x\|}{\|z-x\|}.$$

(The final inequality follows from the facts that $\Big\|z(\|z-x\|-\|z\|)\Big\| = \Big|\|z-x\|-\|z\|\Big| \cdot \|z\|$ and $\Big|\|z-x\|-\|z\|\Big| \leq \|x\|$.) Now, since X is compact, it is bounded, so there exists M such that $\|x\| \leq M$ for all $x \in X$. Combining all inequalities and using the fact that $\|z\| - M \leq \|z\| - \|x\| \leq \|z-x\|$, we then have that

$$\left\| u_z(x) - \frac{-z}{\|z\|} \right\| \leq \frac{2M}{\|z-x\|} \leq \frac{2M}{\|z\| - M}$$

implying that as $\|z\| \to \infty$, $\|u_z(x) - \frac{-z}{\|z\|}\| \to 0$. That is, as the magnitude of z increases without bound, $u_z(x)$ approaches $\frac{-z}{\|z\|}$.

Therefore, there exists M' such that for all z with $\|z\| \geq M'$, $\|\frac{x-z}{\|x-z\|} - \frac{-z}{\|z\|}\| < 1/2$. Now, note that $W_2(X,z) = \#\{x : \frac{x-z}{|x-z|} = y\} \pmod 2$ for any regular value $y \in S^{n-1}$. Let $y = \frac{z}{|z|}$, and see that y is a regular value: Since for all z with $\|z\| \geq M'$, $u_z(x) \in B(\frac{-z}{\|z\|}, \frac{1}{2}) \cap S^{n-1}$, we have that $y \notin \mathrm{Im}(u_z)$.[1] Hence, y is trivially a regular value and clearly $\#\{x : \frac{x-z}{\|x-z\|} = y\} = 0$, so we have that $W_2(X,z) = \#\{x : \frac{x-z}{\|x-z\|} = y\} \pmod 2 = 0$.

11. **Prove the Jordan-Brouwer Separation Theorem: The complement of the compact connected hyper surface (i.e. codimension 1 submanifold) X in $\mathbb{R}^n$ consists of two connected open sets, the "outside" D_0 and the "inside" D_1. Moreover, $\bar{D_1}$ is a compact manifold with boundary $\partial\bar{D_1} = X$.**

    We know by part (9) that the complement of X in $\mathbb{R}^n$ consists of precisely two connected components, D_0 and D_1. Note that since X is closed, $\mathbb{R}^n - X$ is open. Since D_0 and D_1 are connected components of $\mathbb{R}^n - X$, we can take open sets U_0, U_1 of $\mathbb{R}^n$ so that $D_0 \subset U_0, D_1 \subset U_1$. Then $D_0 = U_0 \cap \mathbb{R}^n - X$, so as an intersection of open sets, D_0 is open. By a similar argument D_1 is open as well.

    Now we seek to show that $\bar{D_1}$ is a compact manifold with boundary $\partial\bar{D_1} = X$. By part (10) we know that $\forall z$ sufficiently large, $W_2(X,z) = 0$ which implies that $z \in D_0$. Thus D_1 is bounded. By taking the closure

[1] Here $B(x,\epsilon)$ is the ball of radius $\epsilon > 0$ centered at x.

of D_1, we have constructed a closed and bounded subset of $\mathbb{R}^n$ which implies that $\bar{D_1}$ is compact. To show that $\bar{D_1} = D_1 \cup X$, note that D_0 is open $\Rightarrow D_0^c = D_1 \cup X$ is closed. Thus, $\bar{D_1} \subset D_1 \cup X$. Now we claim that $D_1 \cup X \subset \bar{D_1}$ or more specifically that $X \subset \bar{D_1}$. Pick an $x \in X$ and $z \in D_1$. We will construct a sequence $(z_n) \in D_1$ that converges to x. By exercise (4) we can take any arbitrarily small neighborhoods U of x and there exists a $z_i \in U$ such that z_i is connected to z. As $z \in D_1$, and D_1 is a connected component of $\mathbb{R}^n \setminus X$, it must be that $z_i \in D_1$ as well. Since our $\mathbb{R}^n$ is second countable, from a previous homework, if we construct our sequence with nested open sets about x we can form a sequence (z_n) in D_1 that converges to x. Since our choice of x was arbitrary, we have that $X \subset \bar{D_1}$. Thus $\bar{D_1} = D_1 \cup X$ and $\partial\bar{D_1} = X$.

Now we show that $\bar{D_1}$ is a manifold with boundary. Let $x \in X$. Since X is a submanifold of $\mathbb{R}^n$ of dimension $n-1$, by the local immersion theorem, by considering the inclusion map from X to $\mathbb{R}^n$ we can find a parametrization $\psi : B \to U$, where B is an open ball around 0 in $\mathbb{R}^n$ and U is a neighborhood of x in $\mathbb{R}^n$, so that if $\psi = (x_1, x_2, ..., x_n)$ then $\psi|_{U\cap X} = (x_1, x_2, ..., x_{n-1}, 0)$. That is, $\psi(B \cap \mathbb{R}^{n-1}) = X \cap \psi(B)$.

We claim that U is separated into D_0, D_1 by X. Pick $z \in \mathbb{R}^n - X$. Then by exercise (4) it can be joined by a path to some element in U. Thus these two elements belong to the same connected component and thus by exercise (6) have the same winding number. So both of these elements belong to either D_0 or D_1. Since this holds for every element in $\mathbb{R}^n - X$ and D_0, D_1 are nonempty, we have that U is separated into subsets of D_0 and D_1 by X.

Thus consider $\psi(B \cap (H^n - \mathbb{R}^{n-1}))$ and $\psi(B \cap (-H^n - \mathbb{R}^{n-1}))$. Since $B \cap (H^n - \mathbb{R}^{n-1})$ is connected and ψ is continuous, $\psi(B \cap (H^n - \mathbb{R}^{n-1}))$ is connected. So it maps entirely into either $U \cap D_0$ or $U \cap D_1$. By symmetry, $\psi(B\cap(-H^n-\mathbb{R}^{n-1}))$ is also connected and maps entirely into either D_0 or D_1. Because ψ is a diffeomorphism, it is a bijection. Thus it must be that $\psi(B \cap (H^n - \mathbb{R}^{n-1})) \cap \psi(B \cap (-H^n - \mathbb{R}^{n-1})) = \emptyset$ for if this wasn't the case, then our map ψ would fail to satisfy surjectivity. So we can assume without loss of generality that $\psi(B\cap(H^n-\mathbb{R}^{n-1})) \subset D_1$ and $\psi(B \cap (-H^n - \mathbb{R}^{n-1})) \subset D_0$. Thus by restricting ψ to $B \cap H^n$, we have constructed a parametrization of $\bar{D_1}$, which shows that $\bar{D_1}$ is a manifold with boundary. (If $\psi(B\cap(-H^n-\mathbb{R}^{n-1})) \subset D_1$, then one could just compose ψ with the function that takes the last coordinate of an element in $\mathbb{R}^n$ to its negative to construct the parametrization.)

12. **Given $z \in \mathbb{R}^n \backslash X$, let r be any ray emanating from z that is transversal to X. Show that z is inside X if and only if r intersects X in an odd number of points.**

We first consider $r \cap X$. Note that r is a 1-dimensional manifold. Since r is transversal to X, $r \cap X$ is a manifold with with codimension

$$\text{codim } r + \text{codim } X$$

However, $\dim r = 1$ and $\dim X = (n-1)$, so codim $r = n-1$ and codim $X = 1$. Thus $r \cap X$ has codimension $n - 1 + 1 = n$, and $r \cap X$ has dimension 0. However, r is closed and X is compact, so $r \cap X$ is compact. Since $r \cap X$ is 0-dimensional and compact, it must be finite.

Assume z is inside X. Let $m = \#r \cap X$ be the number of times r intersects X. By exercise 10, there exists some M_0 such that for $y \in r$, $|y| > M_0$, we have that $W_2(X, y) = 0$. By the above, we have that m is finite, so we can let M_1 be the distance of the farthest intersection between r and X from z. Thus take $y \in r$ so that $|y| > M_0$ and $d(y, z) > M_1$. Note that since $d(y, z) > M_1$, we have that

$$l := \text{number of times } r \text{ intersects } X \text{ between } z \text{ and } y$$

is equal to m. Thus by exercise 8, we have that

$$\begin{aligned} W_2(X, z) &= W_2(X, y) + l \mod 2 \\ &= W_2(X, y) + m \mod 2 \end{aligned}$$

However, we had that $W_2(X, y) = 0$ by our definition of y, so

$$W_2(X, z) = m \mod 2 \quad (1)$$

.

We prove the forward direction. Suppose z is inside X. Then $W_2(X, z) = 1$, so thus $m \equiv 1 \mod 2$, i.e. m is odd, as desired.

The backward direction is similar. Suppose $m \equiv 1 \mod 2$. Then $W_2(X, z) = 1$ by equation (1), so z is inside X by definition.