MATH 141: Differential Topology
Homework #2
Due September 18, 2014

Please turn in the starred (*) problems only.

1. Show that $f : X \to Y$ is continuous if and only if the inverse image of every closed set is closed.

2. Show that if A and B are subsets of a topological space X (with topology $\mathcal{T}$) such that A and B are compact, then $A \cup B$ is compact. If X is also Hausdorff, show that $A \cap B$ is compact.

3.* Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. [1]

4.* Let $(X, \mathcal{T}_X)$ be a topological space and $(Y, \mathcal{T}_Y)$ be a topological space that is Hausdorff. For a function $f : X \to Y$, denote by Γ_f, the graph of f, the set

$$\Gamma_f = \{(x, y) \in X \times Y : y = f(x)\}.$$

Show that if f is continuous, then Γ_f is closed in the product topology on $X \times Y$. [2]

5.* Show that if $(X, \mathcal{T})$ is second-countable and $S \subset X$, then every limit point of S is a limit of a sequence in S. [3]

[1] Hint: Read the section in Hatcher on cut points. Note that the discussion in the notes is not a proof, and you will need to formalize the argument.

[2] Hint #1: Show that Γ_f^C is open. Hint #2: For $(x, y) \in \Gamma_f^C$, find open $y \in V \subset Y$ with $f(x) \notin V$. Then use continuity to find open $U \subset X$ so that $U \times V$ is disjoint from Γ_f.

[3] Hint: Note that for any open set U that contains x, there is a basis set V such that $x \in V \subset U$. Use second-countability and this fact to construct a sequence that converges to x, if x is a limit point.