Math 104-1
Midterm 1
September 21, 2015

Name: ______________________________

- You will have **50 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

> On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ______________________________

1. Determine whether the following statements are true or false. No justification is required.

   (a) (2 points) The set $\mathbb{Q}$ of rational numbers is countably infinite.

   TRUE false

   (b) (2 points) The set $\mathbb{R}$ of real numbers is countably infinite.

   true **FALSE**

   (c) (2 points) Every monotone sequence has a limit.

   TRUE false

   (d) (2 points) Suppose that $f : X \to Y$ be a function and $A, B \subset X$. Then, $f(A \cap B) = f(A) \cap f(B)$.

   true **FALSE**

2. Define the following terms.

   (a) (3 points) $\limsup_{n\to\infty} s_n$

   Solution:
   $$\limsup_{n\to\infty} = \lim_{N\to\infty} \sup\{s_n : n > N\}.$$

   (b) (3 points) preimage (or inverse image)

   Solution: Let $f : X \to Y$ and $C \subset Y$. Then the preimage of C under f is $f^{-1}(C) = \{x \in X : f(x) \in C\}$.

   (c) (3 points) decreasing sequence

   Solution: A sequence $(s_n)_{n=1}^{\infty}$ is a decreasing sequence if for every n, $s_{n+1} \leq s_n$.

   (d) (3 points) injective function

   Solution: A function $f : X \to Y$ is injective if $f(x) = f(y)$ implies that $x = y$.

3. (10 points) Let $S \subset \mathbb{R}$ be a non-empty subset, and suppose $u \in \mathbb{R}$ is a lower bound for S. Suppose also that $u \neq \inf S$. Show that there exists a rational number $r > u$ such that r is also a lower bound for S.

Solution: Let u be a lower bound for S. By the corollary to the Completeness Axiom (Corollary 4.5 in Ross), then S has a greatest lower bound $\inf S$. Since $\inf S$ is the greatest lower bound for S, we have that $u \leq \inf S$. The assumption that $u \neq \inf S$ then implies that $u < \inf S$. By the density of the rationals (Theorem 4.7 in Ross), there exists a rational number r such that $u < r < \inf S$.

If $s \in S$, then we have that $r < \inf S \leq s$ since $\inf S$ is a lower bound for S. Thus, we conclude that $r > u$ is a lower bound for S.

4. (10 points) Suppose $(s_n)_{n=1}^{\infty}$ is a convergent sequence, $N_0 \in \mathbb{N}$, and $(t_n)_{n=1}^{\infty}$ is a sequence such that $s_n = t_n$ for all $n > N_0$. Show that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n.$$

Solution: Since $(s_n)_{n=1}^{\infty}$ converges, we have an $s \in \mathbb{R}$ such that $\lim_{n\to\infty} s_n = s$.

Let $\epsilon > 0$. By definition of the limit, this implies that there exists N_1 such that $n > N_1$ implies $|s_n - s| < \epsilon$. Set $N = \max\{N_0, N_1\}$. Then, if $n > N$, we have that $n > N \geq N_0$, so that $|t_n - s| = |s_n - s|$. Moreover, $n > N \geq N_1$ implies that $|s_n - s| < \epsilon$. Hence, we conclude that for all $n > N$, $|t_n - s| < \epsilon$.

Thus, $\lim_{n\to\infty} t_n = s = \lim_{n\to\infty} s_n$.

5. (10 points) Suppose that $(s_n)_{n=1}^\infty$ is a convergent sequence. Show that $(s_n)_{n=1}^\infty$ is Cauchy. (Do not simply refer to the theorem that states this result.)

Solution: Since $(s_n)_{n=1}^\infty$ converges, there exists $s \in \mathbb{R}$ such that $\lim_{n\to\infty} s_n = s$. Let $\epsilon > 0$. By definition of the limit, then there exists an N such that $n > N$ implies that $|s_n - s| < \frac{\epsilon}{2}$.

Now, suppose $m, n > N$. We have that

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m|$$

by the triangle inequality. Moreover, we know that $|s - s_m| = |s_m - s|$, and since $n, m > N$, we have $|s_n - s| < \epsilon/2$ and $|s_m - s| < \epsilon/2$. Hence, we have that for every $\epsilon > 0$, there exists an N such that $n, m > N$ implies,

$$|s_n - s_m| \leq |s_n - s| + |s_m - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, by definition, we conclude that $(s_n)_{n=1}^\infty$ is Cauchy.

(This space intentionally left blank.)

Question:	1	2	3	4	5	Total
Points:	8	12	10	10	10	50
Score:						