Math 104 Section 1
Final Exam
December 17, 2015

Name: ______________________________

- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.

- Unless explicitly stated otherwise, you may assume any derivatives or integrals of functions that you learned in calculus. These include common functions like x^n, $\sin x$ and $\cos x$, e^x, $\log x$, etc.

- You are allowed one page (one side only) of your own personal notes. Books and other notes are not permitted. No electronic devices, including cellphones, headphones, calculation aids, will be permitted for any reason. Optical aids that are non-prescription will also not be permitted. Please staple your notes to your exam before turning it in. Do not write on the backside of your notes, even for scratch work.

- You will have **160 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.

- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.

- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ______________________________

1. Determine whether the following statements are true or false. No justification is required.

   (a) (2 points) If $f_n \to f$ uniformly on $[a,b]$ and f_n is integrable on $[a,b]$ for all n, then $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.

   TRUE false

   (b) (2 points) If $f_n \to f$ uniformly on $[a,b]$ and f_n is differentiable on (a,b) for all n, then $\lim_{n\to\infty} f_n'(x) = f'(x)$ for all $x \in (a,b)$.

   true **FALSE**

   (c) (2 points) Let $f : \mathbb{R} \to \mathbb{R}$. If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$, then f is continuous at a.

   TRUE false

   (d) (2 points) If the Taylor series for a function $f : \mathbb{R} \to \mathbb{R}$ centered at 0 has infinite radius of convergence, then the Taylor series converges to f on $\mathbb{R}$.

   true **FALSE**

   (e) (2 points) Let (X,d) be a metric space, and $\{U_i\}_{i\in I}$ a collection of open sets. The intersection $\cap_{i\in I} U_i$ is open.

   true **FALSE**

   (f) (2 points) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

   true **FALSE**

2. Define the following terms.

(a) (2 points) metric space

Solution: A metric space is pair (X, d) where X is a set, and $d : X \times X \to \mathbb{R}$ is a function such that,

1. $d(x, y) \geq 0$ for all $x, y \in X$,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, y) = d(y, x)$ for all $x, y \in X$,
4. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

(b) (2 points) absolutely convergent series

Solution: A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

(c) (2 points) inverse image (or preimage)

Solution: Let $f : X \to Y$ be a function and $S \subset Y$. Then the inverse image of S (under f) is $f^{-1}(S) = \{x \in X : f(x) \in S\}$.

(d) (2 points) uniform continuity

Solution: A function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on S if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $x, y \in S$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$.

3. (10 points) Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ and that $\limsup_{n\to\infty} x_n > a$. Show that there are infinitely many n for which $x_n > a$.

Solution: Suppose $\limsup_{n\to\infty} x_n > a$. This means that $\lim_{N\to\infty} \sup\{x_n : n > N\} > a$. Assume $\limsup x_n$ is finite, and denote $s = \limsup_{n\to\infty} x_n$. Letting $\epsilon = s - a > 0$, by the definition of the limit, there exists an M such that $N > M$ implies $|\sup\{x_n : n > N\} - s| < \epsilon$. This means that for $N > M$, we have $\sup\{x_n : n > N\} > s - \epsilon = a$.

Suppose there are only finitely many n for which $x_n > a$. Let n_0 be the largest such n (with the convention that $n_0 = 0$ if there are no such n), and set $N = \max\{n_0, M\}+1$, so that $N > n_0$ and $N > M$. Since $N > M$, we have that $\sup\{x_n : n > N\} > a$, which means that there must exist an $n_1 > N$ such that $x_{n_1} > a$. But then $n > N > n_0$, which violates that n_0 was the largest value of n for which $x_n > a$.

Hence, there must be infinitely many such n.

In the case where $\limsup x_n = \infty$, then by definition, there exists M such that $N > M$ implies that $\sup\{x_n : n > N\} > a + 1$, and the above argument still holds.

4. (10 points) Suppose that (X, d) is a complete metric space and $(x_n)_{n=1}^{\infty}$ is a sequence in X such that $d(x_n, x_m) \leq \max\{\frac{1}{n}, \frac{1}{m}\}$ for all n, m. Show that $(x_n)_{n=1}^{\infty}$ converges.

Solution: Since (X, d) is a complete metric space, every Cauchy sequence in X converges. Thus, we will show that (x_n) is a Cauchy sequence.

Let $\epsilon > 0$. Set $N = \frac{1}{\epsilon}$. Then, if $n, m > N$, we have that $\frac{1}{n} < \frac{1}{N} = \epsilon$ and $\frac{1}{m} < \frac{1}{N} = \epsilon$, so that

$$d(x_n, x_m) \leq \max\{\frac{1}{n}, \frac{1}{m}\} < \epsilon.$$

Hence, (x_n) is a Cauchy sequence, and (x_n) converges.

5. (10 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and that f' is bounded on $\mathbb{R}$. Show that there exists a constant C such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}$. *(Hint: Use the Mean Value Theorem.)*

Solution: Since f' is bounded on $\mathbb{R}$, there exists a C such that $|f'(x)| \leq C$ for all $x \in \mathbb{R}$.

Suppose $x, y \in \mathbb{R}$, and without loss of generality, assume $x < y$ (if $x = y$, then $|f(x) - f(y)| = 0 \leq C|x - y| = 0$). Since f is differentiable on $\mathbb{R}$, it is continuous on $\mathbb{R}$ (Ross Theorem 28.2). By the Mean Value Theorem on $[x, y]$ there exists $c \in (x, y)$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(c).$$

Taking absolute values and multiplying by $|x-y|$ yields $|f(x)-f(y)| = |f'(c)||x-y| \leq C|x - y|$, as desired.

6. (10 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is integrable on $[a, b]$ and $a < c < b$. Show that f is integrable on $[a, c]$.

Solution: Let $\epsilon > 0$. By Theorem 32.5, since f is integrable on $[a, b]$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

Let $P' = P \cup \{c\}$, so that $P \subseteq P'$. By Lemma 32.2, we have that $L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P)$, and it follows that $U(f, P') - L(f, P') \leq U(f, P) - L(f, P) < \epsilon$.

Write P' as $P' = \{a = t_0 < t_1 < \cdots < t_k = c < t_{k+1} < \cdots < t_n = b\}$. Then,

$$U(f, P') - L(f, P') = \sum_{j=1}^{n} [M(f, [t_{j-1}, t_j]) - m(f, [t_{j-1}, t_j])](t_j - t_{j-1}).$$

Each term in the summation is non-negative, since $M(f, [t_{j-1}, t_j]) - m(f, [t_{j-1}, t_j]) \geq 0$ and $t_j - t_{j-1} > 0$. Thus, if we take the sum of the first k terms, we obtain

$$\sum_{j=1}^{n} [M(f, [t_{j-1}, t_j]) - m(f, [t_{j-1}, t_j])](t_j - t_{j-1}) \geq \sum_{j=1}^{k} [M(f, [t_{j-1}, t_j]) - m(f, [t_{j-1}, t_j])](t_j - t_{j-1}).$$

Take Q to be the partition $\{a = t_0 < t_1 < \cdots < t_k = c\}$ of $[a, c]$. The right hand side of the above equation is $U(f, Q) - L(f, Q)$, and the left hand side is $U(f, P') - L(f, P')$, which we have already show is less than ϵ. Hence, $U(f, Q) - L(f, Q) < \epsilon$, so by Theorem 32.5, f is integrable on $[a, c]$.

7. (10 points) Show that

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{2^n(1+x^{2n})}$$

converges uniformly to a continuous function on $\mathbb{R}$.

Solution: We note that x^{2n} is non-negative for any positive integer n, so that $|x^{2n}| < |1 + x^{2n}|$. This implies that

$$\left|\frac{x^{2n}}{2^n(1+x^{2n}}\right| = \frac{1}{2^n}\left|\frac{x^{2n}}{1+x^{2n}}\right| < \frac{1}{2^n}.$$

Setting $M_n = \frac{1}{2^n}$ and $g_n(x) = \frac{x^{2n}}{2^n(1+x^{2n})}$, we have that $|g_n(x)| < M_n$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} M_n$ is a geometric series with common ratio $\frac{1}{2}$, which has absolute value less than 1, the geometric series converges. By the Weierstrass M-test, it follows that

$$\sum_{n=1}^{\infty} g_n(x)$$

converges uniformly. Each $g_n(x)$ is a continuous function on $\mathbb{R}$, and the uniform limit of continuous functions is continuous (Ross Theorem 24.3), so

$$\sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n(1+x^{2n})}$$

converges uniformly to a continuous function.

8. (10 points) Let (X, d) be a metric space and $(x_n)_{n=1}^{\infty}$ be a convergent sequence in X. Let $x = \lim_{n\to\infty} x_n$ and $K = \{x_n : n \in \mathbb{N}\} \cup \{x\}$. Show that K is a compact subset of X.

Solution: Let $\{U_i\}_{i\in I}$ be any open cover of K. Since $K \subseteq \cup_{i\in I} U_i$, there exists some $i_0 \in I$ for which $x \in U_{i_0}$. We have that U_{i_0} is an open neighborhood of x, so by Lebl Proposition 7.3.8, there exists an N such that $n > N$ implies that $x_n \in U_{i_0}$. As $\{U_i\}_{i\in I}$ is an open cover of K, for each $x_n, n \leq N$, there exists a $U_{i_n} \in \{U_i\}_{i\in I}$ so that $x_n \in U_{i_n}$.

We claim that $\{U_{i_0}, U_{i_1}, \ldots, U_{i_N}\}$ is a finite subcover of $\{U_i\}_{i\in I}$. We can see it is a finite subset of the original cover, so it suffices to show that every element of K is in one of the $U_{i_j}, j = 1, \ldots, N$.

If $n > N$, then we have that $x_n \in U_{i_0}$. If $n \leq N$, then $x_n \in U_{i_n}$. And $x \in U_{i_0}$, so every element of K is in the union of the U_{i_j}.

9. (a) (10 points) Let (Y, d_Y) be a metric space with $d_Y : Y \times Y \to \mathbb{R}$ the discrete metric on Y. Show that every subset of Y is both open and closed.

Solution: Notice that for any $y \in Y$, $B(y, 1) = \{y\}$. This is true because $d_Y(y, y) = 0 < 1$, but if $z \neq y$, then $d_Y(y, z) = 1$, so $z \notin B(y, 1)$.
Let $S \subset Y$ be any subset. Then, for any $y \in S$, $B(y, 1) = \{y\} \subset S$, so S is open by definition.
Similarly, S^C is open (as we just showed that any subset is open). Hence, S is closed.

(b) (10 points) Let (X, d_X) be a connected metric space, and (Y, d_Y) a discrete metric space as in part (a). Show that $f : X \to Y$ is continuous if and only if f is a constant function (i.e. there exists a $c \in Y$ so that for every $x \in X$ we have that $f(x) = c$).

Solution: ($\Leftarrow$) Suppose that f is a constant function. Then, given $\epsilon > 0$, we can set $\delta = 1$. If x_1, x_2 are any points in X, we have that $f(x_1) = f(x_2)$, so that $d_Y(f(x_1), f(x_2)) = 0 < \epsilon$. Hence, $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \epsilon$, as desired.

($\Rightarrow$) Suppose that f is continuous. We will assume that f is not constant and derive a contradiction. Suppose that f is not constant, then there exists $x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$. By part (a), $U_1 = \{f(x_1)\}$ and $U_2 = \{f(x_1)\}^C$ are open. Lebl Theorem 7.5.7 states that $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open. Since $f(x_1) \in U_1$, we have that $x_1 \in f^{-1}(U_1)$, and since $f(x_2) \in U_2$, we have that $x_2 \in f^{-1}(U_2)$. Moreover, we know from Lebl Proposition 0.3.15 that $f^{-1}(U_2) = f^{-1}(U_1^C) = f^{-1}(U_1)^C$. Thus, as $f^{-1}(U_1)^C$ is open, $f^{-1}(U_1)$ is also closed (we already showed $f^{-1}(U_1)$ is open).

But now, $x_1 \in f^{-1}(U_1)$ and $x_2 \notin f^{-1}(U_2)$, so that $f^{-1}(U_1)$ is an open and closed subset of X that is not the empty set nor X. Hence, X is not connected.

This contradicts the assumption that X is connected, so it must be that f is constant.

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Question:	1	2	3	4	5	6	7	8	9	Total
Points:	12	8	10	10	10	10	10	10	20	100
Score:										