Math 104 Section 2
Midterm 2
November 1, 2013

Name: ____________________

- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- Unless explicitly stated otherwise, you may assume any derivatives or integrals of functions that you learned in calculus. These include common functions like x^n, $\sin x$ and $\cos x$, e^x, $\log x$, etc.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- You will have **50 minutes** to complete the exam. The start time and end time will be signaled by the instructor.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ____________________

1. (a) (5 points) Show that if a and b are nonnegative, then $\sqrt{ab} \leq a + b$.

Solution: Suppose $a \geq b$. Then, $\sqrt{ab} \leq \sqrt{a^2} = a \leq a+b$ since b is nonnegative. Similarly, if $b \geq a$, then $\sqrt{ab} \leq \sqrt{b^2} = b \leq a + b$ since a is nonnegative.
In either case, $\sqrt{ab} \leq a + b$.

(b) (10 points) Show that if $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges.

Solution: Since $\sum a_n$ and $\sum b_n$ converge, then so does $\sum a_n + b_n$ since the sum of two convergent series is always convergent.
Now, by part (a), $\sqrt{a_n b_n} \leq a_n + b_n$ since a_n and b_n are nonnegative. Since both sides are nonnegative, we can apply the Comparison Test (Theorem 14.6) to conclude that $\sum \sqrt{a_n b_n}$ is convergent.

2. Determine whether the following statements are true or false. No justification is required.

(a) (5 points) If (s_n) is a sequence and $\liminf s_n < x < \limsup s_n$, then there exists a subsequence of (s_n) converging to x.

true **FALSE**

> **Solution:** If we let $s_n = (-1)^n$, then $\limsup s_n = 1$ and $\liminf s_n = -1$, but s_n has no other subsequential limits.

(b) (5 points) If $\sum a_n$ is a convergent series, then every rearrangement $\sum a_{\sigma(n)}$ converges to $\sum a_n$.

true **FALSE**

> **Solution:** If $\sum a_n$ is not absolutely convergent (e.g. $a_n = (-1)^n \frac{1}{n}$), then there exists rearrangements that converge to any number x.

(c) (5 points) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$, then f' is continuous on $\mathbb{R}$.

true **FALSE**

> **Solution:** $f(x) = x^2 \sin x$ for $x \neq 0$ and $f(0) = 0$ is an example where f is differentiable everywhere but f' is not continuous at 0 (see Exercise 28.4).

(d) (5 points) Let f be the function such that $f(x) = x$ if x is rational and $f(x) = 0$ otherwise. Then f is integrable on $[0, 1]$.

true **FALSE**

> **Solution:** This is similar to the example $f(x) = 1$ if x is rational and $f(x) = 0$ otherwise. The upper integral will be $\frac{1}{2}$ while the lower integral will be 0 since the supremum $M(f, [t_{k-1}, t_k])$ on any subinterval $[t_{k-1}, t_k]$ will be t_k.

(e) (5 points) If f is continuous on (a, b), then f is integrable on $[a, b]$.

true **FALSE**

> **Solution:** f can be continous on (a, b) but not continuous on $[a, b]$. Take, for example $f(x) = \frac{1}{x}$ on $(0, 1]$ and $f(0) = 0$. Then f is unbounded, so it is not integrable.

3. (20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function (i.e. continuous at every $x_0 \in \mathbb{R}$) and let $(s_n)_{n=1}^{\infty}$ be a sequence with $\limsup s_n = s < \infty$. Show that $\limsup[f(s_n)] \geq f(s)$.

Solution: Let $(s_n)_n$ be a sequence. By Theorem 11.7, there exists a subsequence $(s_{n_k})_k$ whose limit is $s = \limsup s_n$. Since f is continuous, then $\lim_{k\to\infty} f(s_{n_k}) = f(s)$. But then $(f(s_{n_k}))_k$ is a subsequence of $(f(s_n))_n$ that converges to $f(s)$. By Theorem 11.8, $\limsup[f(s_n)]$ is the supremum of all subsequential limits of $(f(s_n))_n$, so $\limsup[f(s_n)] \geq f(s)$.

4. (a) (20 points) Prove that $|\sin x - \sin y| \le |x - y|$ for all $x, y \in \mathbb{R}$.

Solution: Let $x, y \in \mathbb{R}$ and without loss of generality, assume $x < y$ (if $x = y$, then both sides are 0, so this is trivial). Since $f(x) = \sin x$ is differentiable, by the Mean Value Theorem (Theorem 29.3), there exists a $z \in [x, y]$ such that $f'(z) = \frac{f(x)-f(y)}{x-y} = \frac{\sin x - \sin y}{x-y}$. But also, $f'(z) = \cos z$ and $|\cos z| \le 1$. Hence, we have that

$$\left|\frac{\sin x - \sin y}{x - y}\right| = |\cos z| \le 1.$$

This yields $|\sin x - \sin y| \le |x - y|$ as desired.

(b) (20 points) Show that $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

Solution: Let $\epsilon > 0$. By part (a), if we let $\delta = \epsilon$, then $|x - y| < \delta$ implies that

$$|f(x) - f(y)| = |\sin x - \sin y| \leq |x - y| < \delta = \epsilon.$$

Hence, f is uniformly continuous.

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Question:	1	2	3	4	Total
Points:	15	25	20	40	100
Score:					