Math 104 Section 2
Midterm 1 Solutions
September 25, 2013

Name: ______________________________

- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. You may use any result proved in class or the text, but be sure to clearly state the result before using it, and to verify that all hypotheses are satisfied.
- This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- You will have **50 minutes** to complete the exam. The start time and end time will be signaled by the instructor.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ______________________________

1. (16 points) Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: We first show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Suppose $x \in A \cap (B \cup C)$. Then, $x \in A$ and $x \in B \cup C$. The latter implies that $x \in B$ or $x \in C$. Since we already know $x \in A$, this means that $x \in A \cap B$ or $x \in A \cap C$. Therefore, $x \in (A \cap B) \cup (A \cap C)$.

Now we show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Suppose $x \in (A \cap B) \cup (A \cap C)$. Then, $x \in (A \cap B)$ or $x \in (A \cap C)$. In the first case, we have that $x \in A$ and $x \in B$. In the second case, we have $x \in A$ and $x \in C$. In either case, we have $x \in A$. Moreover, we have $x \in B$ or $x \in C$. Hence, $x \in B \cup C$. Since $x \in A$ and $x \in B \cup C$, we have $x \in A \cap (B \cup C)$.

2. Let $f : A \to B$ and $g : B \to C$. Determine whether the following statements are true or false. No justification is required.

   (a) (3 points) If $g \circ f$ is injective, then f is injective.

   TRUE false

   (b) (3 points) If $g \circ f$ is surjective, then f is surjective.

   true **FALSE**

   (c) (3 points) If f is surjective, then $f(A) = B$.

   TRUE false

3. Determine whether the following statements are true or false. No justification is required

   (a) (3 points) A bounded subset of $\mathbb{R} \setminus \mathbb{Q}$ has a supremum in $\mathbb{R} \setminus \mathbb{Q}$.

   true **FALSE**

   (b) (3 points) If (s_n) is a sequence of positive real numbers and $\lim s_n = \infty$, then $\lim \frac{1}{s_n} = 0$.

   TRUE false

   (c) (3 points) Every monotonic sequence in $\mathbb{R}$ that converges to a real number is bounded.

   TRUE false

   (d) (3 points) If (s_n) is a sequence and $\liminf s_n \neq \limsup s_n$, then $\lim s_n$ does not exist.

   TRUE false

   (e) (3 points) Every bounded sequence in $\mathbb{R}$ converges to a real number.

   true **FALSE**

4. (20 points) Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

Solution: The Archimedean property states that if $A > 0$ and $B > 0$, then for some positive integer n, we have $nA > B$.

First, we let $A = 1$ and $B = a$. Then, there exists an $n_1 \in \mathbb{N}$ such that $n_1 \cdot 1 > a$. In other words, $n_1 > a$.

Now let $A = a$ and $B = 1$. Then, there exists $n_2 \in \mathbb{N}$ such that $n_2 \cdot a > 1$. In other words, $a > \frac{1}{n_2}$.

Now let $n = \max\{n_1, n_2\}$. Then, we have $n \geq n_1$ and $n \geq n_2$, so that $a < n_1 \leq n$, and $\frac{1}{n} \leq \frac{1}{n_2} < a$.

Putting these together yields an $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

5. (a) (10 points) Give an example, with brief justification, where $\lim |s_n|$ exists but $\lim s_n$ does not exist.

Solution: Let $s_n = (-1)^n$. We have shown in class that $\lim s_n$ does not exist. However, $|s_n| = 1$, so that $(|s_n|)$ is just the constant sequence where each term is 1. Hence, $\lim |s_n| = 1$.

(b) (10 points) Let $a, b \in \mathbb{R}$. Prove that

$$|a| - |b| \leq |a - b|.$$

Solution: The triangle inequality states that $|A + B| \leq |A| + |B|$.
Let $A = a - b$ and $B = b$. Then,

$$|a| = |(a - b) + b| = |A + B| \leq |A| + |B| = |a - b| + |b|.$$

Subtracting $|b|$ from both sides yields the desired inequality $|a| - |b| \leq |a - b|$.

(c) (20 points) Let (s_n) be a sequence of real numbers such that $\lim s_n = s$. Show that $\lim |s_n| = |s|$.

Solution: Let $\epsilon > 0$. Since $\lim s_n = s$, there exists N such that $n > N$ implies that $|s_n - s| < \epsilon$.
Now consider $||s_n| - |s||$.
From part (b), we know that $|s_n| - |s| \leq |s_n - s|$ by letting $a = s_n$ and $b = s$. But also, letting $a = s$ and $b = s_n$, we find that

$$|s| - |s_n| \leq |s - s_n| = |s_n - s|.$$

If we multiply by -1, we arrive at $-|s_n - s| \leq |s_n| - |s|$. Combine with the first inequality, and we find that

$$-|s_n - s| \leq |s_n| - |s| \leq |s_n - s|.$$

In other words $||s_n| - |s|| \leq |s_n - s|$. So for all $n > N$, we have that

$$||s_n| - |s|| \leq |s_n - s| < \epsilon.$$

This is true for any $\epsilon > 0$, so $\lim |s_n| = |s|$.

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Question:	1	2	3	4	5	Total
Points:	16	9	15	20	40	100
Score:						