Math 104 Section 2

Final Exam

December 19, 2013

Name: ______________________________

- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- Unless explicitly stated otherwise, you may assume any derivatives or integrals of functions that you learned in calculus. These include common functions like x^n, $\sin x$ and $\cos x$, e^x, $\log x$, etc.
- You are allowed one page (front and back) of your own personal notes. Books and other notes are not permitted. No electronic devices, including cellphones, headphones, calculation aids, will be permitted for any reason. Optical aids that are non-prescription will also not be permitted. Please staple your notes to your exam before turning it in.
- You will have **150 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ______________________________

1. (a) (8 points) Define $d_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\bar{d}(x, y) = \sqrt{|x - y|}$. Show that $\bar{d}$ is a metric on $\mathbb{R}$. *Hint: Consider* $(\sqrt{|x-y|} + \sqrt{|y-z|})^2$.

Solution: Since we are dealing with the positive square root, we see that $\bar{d}(x, y) \geq 0$ for all $x, y \in \mathbb{R}$.
Also, if $x = y$, then $\bar{d}(x, y) = \sqrt{|x - y|} = 0$. Conversely, if $\bar{d}(x, y) = \sqrt{|x - y|} = 0$, then $|x - y| = 0$, which implies $x = y$.
Also, $\bar{d}(x, y) = \sqrt{|x - y|} = \sqrt{|y - x|} = \bar{d}(y, x)$.
To prove the triangle inequality, we first note that

$$(\sqrt{|x-y|} + \sqrt{|y-z|})^2 = |x-y| + 2\sqrt{|x-y||y-z|} + |y-z| \geq |x-y| + |y-z|.$$

Taking the square root of both sides, we conclude that $\sqrt{|x-y|} + \sqrt{|y-z|} \geq \sqrt{|x-y| + |y-z|}$. We also note that $|x-z| \leq |x-y| + |y-z|$ implies $\sqrt{x-z} \leq \sqrt{|x-y| + |y-z|}$. Putting all of this together, we have that for $x, y, z \in \mathbb{R}$,

$$\begin{aligned}
\bar{d}(x, z) &= \sqrt{|x-z|} \\
&\leq \sqrt{|x-y| + |y-z|} \\
&\leq \sqrt{|x-y|} + \sqrt{|y-z|} \\
&= \bar{d}(x, y) + \bar{d}(y, z).
\end{aligned}$$

Hence, $\bar{d}$ satisfies the properties of a metric.

(b) (6 points) Show that a subset $U \subseteq \mathbb{R}$ is open in $(\mathbb{R}, \bar{d})$ if and only if U is open in $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$ is the usual metric on $\mathbb{R}$.

> **Solution:** We will differentiate between the open ball in $(\mathbb{R}, \bar{d})$ and $(\mathbb{R}, d)$ by calling them $B_{\bar{d}}(x, R)$ and $B_d(x, R)$. Note that $B_{\bar{d}}(x, R) = B_d(x, R^2)$.
>
> $(\Rightarrow)$ Suppose that U is open in $(\mathbb{R}, \bar{d})$. Then, for each $x \in U$, there exists a $\delta > 0$ such that $B_{\bar{d}}(x, \delta) \subset U$. Then, we have that $B_d(x, \delta^2) \subset U$, so each point in U has a $d-$open ball about it contained in U. Hence, U is open in $(\mathbb{R}, d)$.
>
> $(\Leftarrow)$ Suppose that U is open in $(\mathbb{R}, d)$. Then, for each $x \in U$, there exists a $\delta > 0$ such that $B_d(x, \delta) \subset U$. Then, we also have that $B_{\bar{d}}(x, \sqrt{\delta}) \subset U$. Hence, U is also open in $(\mathbb{R}, \bar{d})$.

(c) (6 points) Show that a sequence $(x_n)_{n=1}^{\infty}$ converges to p in $(\mathbb{R}, \bar{d})$ if and only if it converges to p in $(\mathbb{R}, d)$.

> **Solution:** $(\Rightarrow)$ Suppose (x_n) converges to p in $(\mathbb{R}, \bar{d})$. Let U be any $d-$open neighborhood of p. Then U is also open in $(\mathbb{R}, \bar{d})$ by part (b). Since (x_n) converges to p in $(\mathbb{R}, \bar{d})$, there exists an N such that $n > N$ implies $x_n \in U$ by Proposition 7.3.8. Hence, we have that every neighborhood of p that is open in $(\mathbb{R}, d)$ has an N such that $n > N$ implies $x_n \in U$. By applying Proposition 7.3.8 again for $(\mathbb{R}, d)$, we have that (x_n) converges to p in $(\mathbb{R}, d)$.
>
> $(\Leftarrow)$ The argument is identical to the above, with d and $\bar{d}$ switched.

2. Determine whether the following statements are true or false. No justification is required.

(a) (2 points) If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are convergent sequences and $a_n < b_n$ for all n, then $\lim a_n \leq \lim b_n$.

TRUE false

(b) (2 points) If $f_n \to f$ uniformly, then $f_n \to f$ pointwise.

TRUE false

(c) (2 points) Every continuous function on $[a, b]$ is the uniform limit of polynomials on $[a, b]$.

TRUE false

(d) (2 points) If $E \subseteq X$ is not open, then E is closed.

true **FALSE**

(e) (2 points) Every closed and bounded subset of $\mathbb{R}^3$ is compact.

TRUE false

3. Give examples of the following. No justification is required.

(a) (3 points) A convergent series $\sum a_n$ where a rearrangement $\sum a_{\sigma(n)}$ converges to a different value.

> **Solution:** Any convergent series that is not absolutely convergent will satisfy this property, such as $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$.

(b) (3 points) A metric space (X,d) and a subset $A \subseteq X$ so that $\bar{A} = A$.

> **Solution:** We can take any space (X,d) and any closed subset A. For example, $(\mathbb{R}, d(x,y) = |x-y|)$ and $A = [0,1]$.

(c) (3 points) An open subset of $\mathbb{R}^n$ with the metric $d((x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n)) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2}$.

> **Solution:** The open ball is open in any metric space, so we can pick $B(0,1)$, the open ball of radius 1 centered at the origin.

(d) (3 points) A metric space that is not complete.

> **Solution:** Take $X = (0,1)$ and d the restriction of the usual metric on $\mathbb{R}$ to the open interval $(0,1)$.

(e) (3 points) A bounded function on an interval that is not Riemann integrable.

> **Solution:** Define $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise. Then f is not integrable on $[0,1]$.

4. (10 points) Let f and g be continuous functions on $[a, b]$ and suppose that $f(a) = g(b)$ and $f(b) = g(a)$. Show that there exists some $c \in [a, b]$ such that $f(c) = g(c)$.

Solution: Let $h(x) = f(x) - g(x)$. Without loss of generality, assume $h(a) \geq 0$ (if not, take $-h(x)$). Then,

$$h(b) = f(b) - g(b) = g(a) - f(a) = -h(a) \leq 0.$$

Then, by the intermediate value theorem, since $h(b) \leq 0 \leq h(a)$, we have there exists a $c \in [0, 1]$ such that $h(c) = 0$. In other words, $f(c) - g(c) = 0$, or $f(c) = g(c)$ as desired.

5. (a) (7 points) Show that for any $p > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \sin nx$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx$$

converge uniformly on all of $\mathbb{R}$.

Solution: We will use the Weierstrass M-test. Note that

$$|\frac{1}{n^p} \sin nx| \leq \frac{1}{n^p}$$
$$|\frac{1}{n^p} \cos nx| \leq \frac{1}{n^p}$$

for all $x \in \mathbb{R}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, then by the Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{1}{n^p} \sin nx$ and $\sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx$ converge uniformly on all of $\mathbb{R}$.

(b) (8 points) Show that for any $p > 2$,

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^p} \sin nx\right)' = \sum_{n=1}^{\infty} \frac{1}{n^{p-1}} \cos nx.$$

Solution: If $p > 2$, then $p - 1 > 1$, so by part (a), $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}} \cos nx$ converges uniformly on $\mathbb{R}$. Then, by Theorem 25.2, we can integrate term-by-term, and

$$\int_0^x \sum_{n=1}^{\infty} \frac{1}{n^{p-1}} \cos nt = \sum_{n=1}^{\infty} \frac{1}{n^p} \sin nx.$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^{p-1}} \cos nx &= \left(\int_0^x \sum_{n=1}^{\infty} \frac{1}{n^{p-1}} \cos nt\right)' \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{n^p} \sin nx\right)'. \end{aligned}$$

6. (10 points) Suppose that (X, d) is a complete metric space, and

$$E_1 \supset E_2 \supset E_3 \supset \cdots \supset E_n \supset E_{n+1} \supset ...$$

where each $E_j \subseteq X$ is a nonempty compact subset of X. Show that

$$E = \cap_{j=1}^{\infty} E_j$$

is nonempty.

Solution: Suppose (X, d) is complete and $E = \cap_{j=1}^{\infty} E_j$ an intersection of countably infinite nonempty compact sets $E_1 \supset E_2 \supset E_3 \supset \cdots$. For each E_n take any point $x_n \in E_n$. This gives us a sequence $\{x_n\}_{n=1}^{\infty}$ in E_1. Since E_1 is compact, by Theorem 7.4.8, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to some point $x \in E_1$. But also, for $k > 1$, we have $n_k > 1$, so each $x_{n_k} \in E_2$. Again, E_2 is compact, so there exists a further subsequence $x_{n_{k_j}}$ that converges to a point in E_2. But since subsequences of convergence sequences converge to the same point, it must be that $x_{n_{k_j}}$ converges to x. Hence, $x \in E_2$. We can then proceed inductively, and since each E_i is compact, $x \in E_i$. Hence, $x \in \cap_{j=1}^{\infty} E_j$, so the intersection is nonempty.

7. (a) (10 points) Let d_n be the usual metric on $\mathbb{R}^n$, and d_1 be the restriction to $[0,1]$ of the usual metric on $\mathbb{R}$. Let $p, q \in \mathbb{R}^n$. Show that $f : [0,1] \to \mathbb{R}^n$ defined by $f(t) = (1-t)p + tq$ is continuous.

Solution: If $p = q$, then $f(t) = (1-t)p + tq = p$ for all t, so $f(t)$ is a constant function. For any $\epsilon > 0$, for any $s, t \in [0,1]$, we have that $d_n(f(s), f(t)) = d_n(p,p) = 0 < \epsilon$, so we can pick any $\delta > 0$ and we satisfy the necessary condition for continuity.

Otherwise, let $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ be points in $\mathbb{R}^n$. Let $s \in [0,1]$ and fix $\epsilon > 0$. Set $\delta = \frac{\epsilon}{d_n(p,q)}$. Then, if $d_1(t,s) < \delta$, we have that

$$\begin{aligned}
d_n(f(t), f(s)) &= d_n((1-t)p + tq, (1-s)p + sq) \\
&= \sqrt{[(1-t)p_1 + tq_1 - (1-s)p_1 - sq_1]^2 + \cdots + [(1-t)p_n + tq_n - (1-s)p_n -} \\
&= \sqrt{[(s-t)p_1 + (t-s)q_1]^2 + \cdots + [(s-t)p_n + (t-s)q_n]^2} \\
&= \sqrt{(s-t)^2(p_1 - q_1)^2 + \cdots + (s-t)^2(p_n - q_n)^2} \\
&= |s-t|\sqrt{(p_1 - q_1)^2 + \cdots + (p_n - q_n)^2} \\
&= d_1(s,t) d_n(p,q) \\
&< \frac{\epsilon}{d_n(p,q)} d_n(p,q) = \epsilon.
\end{aligned}$$

Hence, f is continuous at s. Since this is true for any $s \in [0,1]$, f is continuous on $[0,1]$.

(b) (10 points) Define a set $S \subseteq \mathbb{R}^n$ to be *star-convex* if there exists a $p \in S$ such that for each $x \in S$, the straight line segment $L_{p,x}$ between p and x lies entirely within S, i.e. $L_{p,x} \subset S$. Show that if S is star-convex, then S is connected.

Solution: First, note that $L_{p,x} = f([0,1])$. Moreover, $[0,1]$ is connected by Proposition 7.2.14. Since f is continuous by part (a), $f([0,1])$ is connected by Lebl Exercise 7.5.5.

Now let S be a star-convex set and suppose that U, V are open subsets of $\mathbb{R}^n$ such that $(U \cup V) \cap S = S$ and $(U \cap V) capS = \emptyset$. Let p be as in the definition. Without loss of generality, $p \in U$.

Consider $x \in S$. Since S is star-convex, $L_{p,x} \subset S$. In other words, we also have that $(U \cup V) \cap L_{p,x} = L_{p,x}$ and $(U \cap V) \cap L_{p,x} = \emptyset$. Since $L_{p,x}$ is connected and U is nonempty in $L_{p,x}$, we can conclude that $V \cap L_{p,x} = \emptyset$. Therefore, $x \in U$. Since this is true for all $x \in S$, then we can conclude that $V \cap S = \emptyset$ and $S \subset U$. As U and V were arbitrary open sets, this shows that S is connected.

(This space intentionally left blank.)

(This space intentionally left blank.)

Question:	1	2	3	4	5	6	7	Total
Points:	20	10	15	10	15	10	20	100
Score:								