

**Problem 1.** Let  $a \in \mathbb{R}$ . Find the coordinate vector  $[\mathbf{x}]_{\mathcal{B}}$  of  $\begin{bmatrix} a \\ 0 \end{bmatrix} \in \mathbb{R}^2$  relative to the basis  $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$

*Solution.* Let  $P_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$  be the change-of-coordinates matrix from  $\mathcal{B}$  to the standard basis in  $\mathbb{R}^2$ . The inversion formula for  $2 \times 2$  matrices gives:

$$P_{\mathcal{B}}^{-1} = \frac{1}{-1} \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

It follows that  $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \mathbf{x} = \begin{bmatrix} -5a \\ 3a \end{bmatrix}$ . □

Note that we can also find  $[\mathbf{x}]_{\mathcal{B}}$  solving  $P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$  via row reduction.

**Problem 2.** Find a basis for  $\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$ .

*Solution.* Let  $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ . We are asked to find a basis for  $\text{Col } A$ . The pivot columns of  $A$  form a basis for  $\text{Col } A$ .

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We find that the first and second columns of  $A$  are pivot columns. Therefore  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$  is a basis for  $\text{Col } A$ , i.e., the span of the given vectors. □

We could have solved the problem by noting that the first and second columns are linearly independent (observe the third coordinate) and that the third is the sum of the first two.

**Problem 3.** Show that the set  $H$  of polynomials in  $\mathbb{P}_2$  that vanish at 0 is a subspace.

*Solution.* If  $p, q \in H$ , then  $(p + q)(0) = p(0) + q(0) = 0 + 0 = 0$ , so  $p + q \in H$ . Thus,  $H$  is closed under addition. If  $p \in H$  and  $c \in \mathbb{R}$ , then  $(cp)(0) = c \cdot p(0) = c \cdot 0 = 0$ , so  $cp \in H$ . Thus,  $H$  is closed under scalar multiplication. Finally, the zero polynomial is an element of  $H$  since it vanishes everywhere. It follows by definition of ‘subspace’ that  $H$  is a subspace. □

We could have similarly shown that the evaluation map  $T: \mathbb{P}_2 \rightarrow \mathbb{R}$  defined by  $T(p) = p(0)$  is linear. We would then immediately conclude that  $H = \ker T$  is a subspace of  $\mathbb{P}_2$ .

**Problem 4.** Find a basis for  $H$  (above).

*Solution.* A polynomial  $p = a_0 + a_1x + a_2x^2$  is  $H$  iff  $a_0 = 0$ . Thus every polynomial  $p \in H$  is a linear combination of  $x$  and  $x^2$ . Since the set  $\{x, x^2\}$  is clearly linearly independent, it is a basis for  $H$ .  $\square$

In the standard basis  $\{1, x, x^2\}$ , the evaluation map  $T$  has standard matrix  $A = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ . The basis of  $\text{Nul } A$  given by the usual algorithm is  $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ , i.e.  $\{x, x^2\}$ .

**Problem 5.** Let  $A \in \mathbb{R}^{m \times n}$  and  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation such that  $T(\mathbf{x}) = A\mathbf{x}$ . Explain why  $\text{ran } T = \text{Col } A$ .

*Solution.* If  $\mathbf{y} \in \text{ran } T$  then  $\mathbf{y} = T(\mathbf{x}) = A\mathbf{x}$  for some  $\mathbf{x} \in \mathbb{R}^n$ . By definition of matrix multiplication  $\mathbf{y} = A\mathbf{x}$  is a linear combination of the columns of  $A$ , i.e.,  $\mathbf{y} \in \text{Col } A$ . Thus  $\text{ran } T \subseteq \text{Col } A$ .

If  $\mathbf{y} \in \text{Col } A$  then by definition of  $\text{Col } A$ ,

$$\mathbf{y} = x_0\mathbf{a}_0 + \cdots + x_n\mathbf{a}_n = A \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix}$$

for some  $x_0, \dots, x_n \in \mathbb{R}$ . It follows that  $\mathbf{y} = A\mathbf{x} = T(\mathbf{x})$  for some  $\mathbf{x} \in \mathbb{R}^n$ , i.e.,  $\mathbf{y} \in \text{ran } A$ . Thus  $\text{Col } A \subseteq \text{ran } T$ .

The statements  $\text{ran } T \subseteq \text{Col } A$  and  $\text{Col } A \subseteq \text{ran } T$  together imply that  $\text{ran } T = \text{Col } A$ .  $\square$

**Problem 6.** Show that every linear map  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  has a non-trivial kernel.

*Solution.* Let  $A \in \mathbb{R}^{2 \times 3}$  be the standard matrix for  $T$ . The Rank Theorem for  $A$  states that  $\dim \text{Col } A + \dim \text{Nul } A = 3$ . Since  $\text{Col } A = \text{ran } T$  and  $\text{Nul } A = \ker T$ , the Rank Theorem for  $A$  is equivalent to the statement

$$\dim \text{ran } T + \dim \ker T = 3.$$

Since  $\text{ran } T$  is a subspace of  $\mathbb{R}^2$ , it has dimension at most 2. Thus  $\ker T$  should have dimension at least 1. Therefore  $\ker T$  must contain a nonzero vector.  $\square$