Problem 1 (4 points)**.** *Let $c > 0$. Find a function $u(x,t)$, $x \in [0,\pi]$, $t \geq 0$, which satisfies*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial t^2},$$

for $t > 0$, $u(0,t) = 0 = u(\pi,t)$, and the initial conditions $u(x,0) = 2\sin x$, $\frac{\partial u}{\partial t}(x,0) = \sin 3x$.

Solution. The general solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos nct + b_n \sin nct) \sin nx$$

Assuming converges, which will follow from the finiteness of the series we find,

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} (-a_n nc \sin nct + b_n nc \cos nct) \sin nx.$$

Thus $2\sin x = u(x,0) = \sum a_n \sin nx$ and $\sin 3x = \frac{\partial u}{\partial t}(x,0) = \sum b_n nc \sin nx$. We conclude that $a_1 = 2$, $b_3 = \frac{1}{3c}$ and all other coefficients are zero. Therefore, $u(x,t) = 2\cos ct \sin x + \frac{1}{3c}\sin 3ct \sin 3x$. □

Problem 2 (4 points)**.** *The Fourier series of $f(x) = x^2$, $x \in [-1,1]$ can be computed to be*

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi^2 n^2} \cos(\pi n x).$$

(1) *Show that the Fourier series above converges uniformly and explain what this means.*
(2) *Show that $f'(x) \sim -\sum \frac{4(-1)^n}{\pi n} \sin(\pi n x)$.*
(3) *The Fourier series of $f'(x)$ does not converge uniformly. What function does it converge to pointwise?*
(4) *Find the Fourier series of $g(x) = x^3$, $x \in [-1,1]$.*

Proof. (1) The Fourier series of $f(x) = x^2$, $x \in [-1,1]$, converges uniformly because it is continuous on $[-1,1]$ and satisfies $f(-1) = f(1)$, and can therefore be extended to a continuous periodic function on $(-\infty,\infty)$.

This means that for all $\epsilon > 0$, for sufficiently large M,

$$\left| f(x) - \left(\frac{1}{3} + \sum_{n=1}^{M} \frac{4(-1)^n}{\pi^2 n^2} \cos(\pi n x) \right) \right| < \epsilon$$

at *each* $x \in [-1,1]$.

(2) The first and second derivatives of $f(x) = x^2$ are continuous on $[-1,1]$ and are therefore piecewise continuous. It follows that

$$\frac{d}{dx} f(x) \sim \frac{d}{dx}\left(\frac{1}{3}\right) + \sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{4(-1)^n}{\pi^2 n^2}\cos(\pi n x)\right) = -\sum \frac{4(-1)^n}{\pi n}\sin(\pi n x)$$

(3) Pointwise convergence is guaranteed since $f'(x)$ is piece-wise continuous. For $x \in (-1,1)$ continuity implies that the series actually converges to $f'(x)$. On the other hand, the end points $x = 1$ and $x = -1$ should be viewed as the same point on the circle and so $f'(x)$ is in this sense discontinuous there. To find the value of f', there we average limits. Since $f(x)$ is continuous as a function on $[-1,1]$, we have $\lim_{x\to 1} f'(x) = 2$ and $\lim_{x\to -1} f'(x) = -2$. Thus

$$-\sum \frac{4(-1)^n}{\pi n}\sin(\pi n x) \longrightarrow \begin{cases} 2x & x \in (-1,1) \\ 0 & x = -1, 1 \end{cases}$$

(4) The function $f(x) = x^2$ is continuous on $[-1, 1]$ and so is certainly piece-wise continuous. We can therefore integrate it term by term:

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi^2 n^2} \cos(\pi n x)$$

$$\frac{x^3}{3} = \frac{x}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi^3 n^3} \sin(\pi n x)$$

$$x^3 = x + \sum_{n=1}^{\infty} \frac{12(-1)^n}{\pi^3 n^3} \sin(\pi n x)$$

Since we know the Fourier series of $f'(x)$ we know that

$$x \sim -\sum_{n=1}^{\infty} \frac{2(-1)^n}{\pi n} \sin(\pi n x).$$

We conclude that

$$x^3 \sim \sum_{n=1}^{\infty} \left(\frac{12(-1)^n}{\pi^3 n^3} - \frac{2(-1)^n}{\pi n} \right) \sin(\pi n x)$$

□