Problem 1 (3 points)**.** *Solve the initial value problem* $y'' + 9y = 27$, $y(0) = 4$, $y'(0) = 6$.

Solution 1. The auxiliary polynomial $r^2 + 9 = 0$ has roots $\pm 3i$. Thus the solution set of the homogeneous equation $y'' + 9y = 0$ is spanned by $\{y_1, y_2\} \equiv \{\cos(3t), \sin(3t)\}$.

Noting that $27 = 27e^{0t}$ and that 0 is not a root of the auxiliary equation $r^2 + 9 = 0$, we apply the method of undetermined coefficients to the particular solution $y_p = A_0 e^{0t} = A_0$. Plugging $y_p = A_0$ into the given differential equation we find $A_0 = 3$. Thus the general solution is

$$y = y_p + c_1 y_1 + c_2 y_2 = 3 + c_1 \cos(3t) + c_2 \sin(3t),$$

with $c_1, c_2 \in \mathbb{R}$.

We now use the initial conditions to find c_1 and c_2:

$$4 = y(0) = 3 + c_1$$

$$6 = y'(0) = (-3c_1 \sin(3t) + 3c_2 \cos(3t))|_{t=0} = 3c_2$$

Thus, $c_1 = 1$ and $c_2 = 2$, and therefore $y_p = 3 + \cos(3t) + 2\sin(3t)$. □

Solution 2. We can write the given differential equation as $y'' + 9(y - 3) = 0$. Writing $z = y - 3$ we find that z satisfies the *homogenous* equation $z'' + 9z' = 0$ with general solution $z = c_1 \cos(3t) + c_2 \sin(3t)$, $c_1, c_2 \in \mathbb{R}$. Thus,

$$y = 3 + c_1 \cos(3t) + c_2 \sin(3t).$$

Proceed as above. □

Problem 2 (3 points)**.** *Find the general solution of* $(D^2 + 6D + 13)^2 y = 0$.

Proof. We can factor $(D^2 + 6D + 13) = (D - (-3 + 2i))(D - (-3 - 2i))$. Thus, $e^{-3t}\cos(2t)$ and $e^{-3t}\sin(2t)$ are two solutions to the given homogeneous equation. Since the factor $(D^2 + 6D + 13)$ appears twice, $te^{-3t}\cos(2t)$ and $te^{-3t}\sin(2t)$ are also solutions. Thus the general solution is

$$y = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t) + c_3 te^{-3t}\cos(2t) + c_4 te^{-3t}\sin(2t),$$

$c_1, c_2, c_3, c_4 \in \mathbb{R}$. □

Problem 3 (2 points)**.** *Let I be the interval $I = (0, 1)$, and let $f : I \longrightarrow \mathbb{R}$ be a function such that*

$$f^{(5)}(t) + 2f^{(4)}(t) + 3f^{(3)}(t) + 4f''(t) + 5f'(t) + 6f(t) = 0$$

for all $t \in I$ and $f^{(4)}(0) = f^{(3)}(0) = f''(0) = f'(0) = f(0) = 0$. Find f.

Solution. The function $g(t) = 0$ is clearly a solution to the given initial value problem. Since any initial value problem on an interval has a unique solution, it must be that $f(t) = g(t) = 0$ for all $t \in I$. □