Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix}$.

Problem 1. *Find the solution set of $A\mathbf{x} = \mathbf{b}$.*

Solution. The augmented matrix of the equation $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 0 & 1 & 2 & 8 \\ 2 & 1 & 0 & 4 \\ 2 & 2 & 2 & 12 \end{bmatrix}$.

Row reduction gives:

$$\begin{bmatrix} 0 & 1 & 2 & 8 \\ 2 & 1 & 0 & 4 \\ 2 & 2 & 2 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & 1 & 2 & 8 \\ 2 & 2 & 2 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & 1 & 2 & 8 \\ 0 & 1 & 2 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -2 & -4 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the solution of the system of linear equations corresponding to the matrix equation $A\mathbf{x} = \mathbf{b}$ is

$$\begin{cases} x_1 = -2 + x_3 \\ x_2 = 8 - 2x_3 \\ x_3 \text{ free.} \end{cases}$$

The general solution of $A\mathbf{x} = \mathbf{b}$ therefore has the form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Thus the solution set is $\{\mathbf{p} + t\mathbf{v} \mid t \in \mathbb{R}\}$ where

$$\mathbf{p} = \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

□

Problem 2. *Do the columns of A span $\mathbb{R}^3$?*

Solution. The columns of A span $\mathbb{R}^3$ iff A has a pivot position in every row. The calculation above shows that A does not. Thus, the columns of A do not span $\mathbb{R}^3$. □

Problem 3. *Does there exist a vector $\mathbf{y} \in \mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{y}$ is inconsistent.*

Solution. In the problem above we showed that the columns of A do not span $\mathbb{R}^3$ and therefore there is a vector $\mathbf{y} \in \mathbb{R}^3$ which is not a linear combination of the columns of A. However, any vector $\mathbf{y} = A\mathbf{x}$ is by definition of the product $A\mathbf{x}$ a linear combination of the columns of A. We conclude there must be a vector $\mathbf{y}$ that is not equal to $A\mathbf{x}$ for any $x \in \mathbb{R}^3$, i.e., such that $A\mathbf{x} = \mathbf{y}$ is inconsistent. □

Problem 4. *Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions of the equation $A\mathbf{x} = \mathbf{b}$. Is every linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ also a solution?*

Solution. Consider the linear combination $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$. We have that $A\mathbf{x} = A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{b} + \mathbf{b} \neq \mathbf{b}$. Thus, $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ is not a solution. □