NORMAL BUNDLES FOR CODIMENSION 2 LOCALLY FLAT IMBEDDINGS

BY

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1. Introduction

THEOREM A. *Let* M^m *and* Q^{m+2} *be TOP (= topological metrizable) manifolds without boundary,* $m \neq 2$, *and suppose that* M *is a locally flat submanifold of* Q. *Then* M *admits a normal microbundle* ν, *and the germ of* ν *about* M *is unique up to ambient isotopy fixing* M *pointwise. If we are given a normal microbundle* ν_0 *over a neighborhood of a closed set* C *of* M, *then we may choose* $\nu = \nu_0$ *over a smaller neighborhood of* C. *Also given two normal microbundles* ν *and* ν' *that agree near* C *we can find an ambient isotopy* h_t, $0 \leq t \leq 1$, *of* $id \mid Q$ *fixing* M *and a neighborhood of* C *so that* $h_1\nu'$ *coincides with* ν *near* M. *Further this isotopy can be as small as we please: for example if* $\mathcal{U}$ *is a prescribed open covering of* Q, *we can choose* h_t *so that for each point* $x \in Q$, *there exists some* $U \in \mathcal{U}$ *such that* $h_t(x) \in U$ *for all* $t \in [0,1]$.

Various statements can be deduced if $\partial M \neq \phi$ and/or $\partial Q \neq \phi$ where ∂ indicates boundary. Here is one: *The theorem holds true with boundaries allowed, in case* $M \cap \partial Q = \partial M$ *and* $m \neq 2, 3$. The proof follows the usual pattern; i.e., apply the original theorem to the boundary pair $(\partial Q, \partial M)$, extend over a suitable pairwise collar of this boundary, then apply the relative theorem to the interior.

Recall that by Kister's theorem [Kister], microbundles contain locally trivial bundles with fiber euclidean space, and these are unique up to fiber preserving isotopy fixing the zero section; this result of course has a relative version. Locally trivial topological bundles with fiber R^n and zero-section can be described as Steenrod bundles with group TOP(n), the homeomorphisms of R^n fixing the origin. (TOP(n) is given the compact open topology.)

Recall [Kneser] [Friberg] [K-S_2, Essay V end of §5] that our structural group TOP(2) deformation retracts to O(2), which is homeomorphic to $S^0 \times S^1$.

These two facts just recalled permit one to deduce existence and uniqueness theorems for closed normal 2-disc bundles to M^m in Q^{m+2}, $m \neq 2$, exactly analogous to the tubular neighborhood existence and uniqueness theorems of differential topology, or the still better known closed collar existence and uniqueness theorems.†

Our theorem was asserted without the restriction $m \neq 2$ in [K_2], but the proof contained an error (page 419, Step 2) found by Bjorn Friberg. The gap remains, but a proof along the lines attempted in [K_2] remains an attractive and very difficult problem; one wants a proof that works in all dimensions and is elementary in its prerequisites as is the proof of the parallel codimension 1 result of M. Brown, see [K-S_2, Essay I, Appendix A].

The proof given here goes back to spring 1969; it reduces quite rapidly to the codimension 2 Hauptvermutung result announced in [K-S_1]. We adopt the differentiable category as a tool in this article, but this is just a question of minor convenience. Our proof uses handlebody theory together with torus geometry (see [K_1] or [K-S_2, Essay I, §3]) and the furling (or gluing) device from [Si_2, §5]. We do not use surgery and there is really no temptation to do so.

Our proof as given for high dimensions applies to ambient dimension $3 = m + 2$. However, to avoid packing along too many cases we summarize right here the modifications necessary, and mention dimension $3 = m + 2$ no more. An essential point is that every smooth 3-manifold appearing in Diagram 4-a below must be known to contain no fake 3-disk (= compact contractible 3-manifold that is not diffeomorphic to B_3). This follows from the uniqueness theorem for smooth structures in dimension 3; but it can be assured trivially by starting (as we may) in 4.1 with a Q^{m+2} that lies in a coordinate chart, where Alexander's theorem excludes fake 3-disks. The necessary results of 3-dimensional handlebody theory, analogous to those cited in §2, but excluding fake 3-disc, are proved in [E. M. Brown], [Husch and Price] and [Stallings]. Finally, when engulfing is called for, it is to be done using the splitting theorem parallel to 2.1, proved using [Husch and Price].

Bjorn Friberg has (according to a private communication) obtained the result that the group $TOP_{m+2,m}$ of homeomorphisms of R^{m+2} fixing R^m pointwise satisfies $\pi_i(TOP_{m+2,m}/O_2) = 0$ for $i \leq m - 2$, even if $m = 2$. Here O_2 is the subgroup of isometries of R^{m+2} fixing R^m pointwise.

Immersion theory reveals that our Theorem A implies:

THEOREM B. *$\pi_i(TOP_{m+2,m};O_2) = 0$ for $i \leq m \neq 2$.*

PROOF OF B FROM A. To an element $x \in \pi_i(TOP_{m+2,m}; O_2)$ immersion theory associates an immersion $f : R^{m+2} \to R^{m+2}$, equal to the identity on R^m‡ and DIFF

†There is one point that needs attention in following this analogy. If one disc bundle T_1 is included in fiber preserving fashion in the interior of a second T_2, it should be possible to isotop T_1 onto T_2 in a fiber preserving fashion. It is enough to know that the group of homeomorphisms of the annulus $2B^2 - B^2$ respecting ∂B^2 is (weakly) homotopy equivalent to the subgroup respecting every circle $\partial\lambda B^2$, and there are proofs of this, see [Hamstrom] and [Scott].

‡f is therefore an imbedding near R^m.

near ∂B^i, such that the micro-bundle map germ $df| : \tau(R^m)|B^i \to \tau(R^m)|B^i$ represents x in the naturally isomorphic group $\pi_i(TOP'_{m+2,m}; DIFF'_{m+2,m})$ where $CAT'_{m+2,m}$ is the (semi-simplicial) group of germs at the origin of CAT automorphisms of R^{m+2} fixing R^m pointwise.† Theorem A and the subsequent comments (or more directly the Handle Lemma 4.1 below) let one find a regular homotopy of f, fixing R^m and fixing a neighborhood of ∂B^i, to an immersion f' that is DIFF near B^i. The deformation of microbundle maps corresponding to this regular homotopy reveals that $x = 0$. (For this use of immersion theory see [K-S_0], and more specifically [Haefliger] and [Rourke-Sanderson].

In §2 we fix some conventions and recall some facts about splitting and fibering. In §3 we discuss the furling (or gluing) device of [Si_2, §5] without which a proof of Theorem A might not be so elementary. In §4 we reduce Theorem A to a handle problem. In §5 the handle problem is reduced to a torus problem using torus methods involving engulfing and furling. In §6 this torus problem is solved by dint of furling, splitting, fibering and s-cobording.

2. Standard Preliminaries.

CAT is an adjective that according to the context means DIFF (= differentiable) or PL (= piecewise linear) or TOP (= topological).

R^n is the n-fold product of the reals, with the convention that R^k is the subset of R^n defined by $x_{k+1} = \dots = x_n = 0$. The n-torus T^n will be $R^n/8Z^n$. The n-ball of radius r is called rB^n; note that $2B^n$ is an obvious subset of T^n; so also is T^k.

The interior of a subset A of a space B is denoted $\mathring{A}$ (= $B - Cl(B - A)$). The formal interior of a manifold M is denoted intM. We do have equality $intB^n = \mathring{B}^n$ (in R^n). The boundary of a manifold M is denoted ∂M; one has $\partial M = M - intM$.

The words "rel C" mean "fixing the restriction to a neighborhood of C".

We shall use DIFF engulfing; one can readily convert PL engulfing methods (see [Hudson]) using Whitehead C^1 triangulations of the manifolds in question, since any smoothly embedded simplex is linear in a suitable DIFF coordinate chart. Alternatively use the chart by chart engulfing technique of [Newman].

From handlebody theory we shall require the s-cobordism theorem and two more results (for CAT = DIFF).

Splitting Theorem 2.1. [S_1] [Kervaire] *Let N^n, $n \geq 6$, be a CAT n-manifold that is proper homotopy equivalent to $K \times R$ for some connected finite complex K. Suppose given a CAT isomorphism $h' : \partial N \to L' \times R$ where L' is a possibly empty compact CAT manifold. Then L' is the boundary of a compact (n-1)-manifold*

†The isomorphism comes from a pairwise version of Kister's theorem and an elementary argument with differentials.

L, and there exists a CAT isomorphism $h : N \to L \times R$ *extending* h', *provided an obstruction is zero in the projective class group* $\tilde{K}_0 Z[\pi_1 N]$. □

FIBERING THEOREM 2.2. [Farrell] *Let* $f : N^n \to K \times S^1$ *be a homotopy equivalence where* K *is a finite complex and* N *is a CAT n-manifold,* $n \geq 6$. *Suppose that* $p_2 f \mid \partial N$ *is a CAT locally trivial bundle map* $\partial N \to S^1$. *Then* $p_2 f$ *is homotopic rel* ∂N *to a CAT locally trivial bundle map* $N^n \to S^1$, *provided an obstruction is zero in the Whitehead group* $Wh(\pi_1(N))$. □

In all our applications the fundamental group will be that of a torus, (i.e., free abelian), in which case the projective class group and the Whitehead group are zero [Bass, Heller, and Swan]. Thus the obstructions above always vanish.

The splitting theorem follows from the s-cobordism theorem and the main result of $[S_1]$ for putting a boundary L on N, extending L'. The basic ingredients of the fibering theorem (for π_1 free abelian at any rate) are the same plus some imaginative geometry [Farrell].

3. UNFURLING AND FURLING.

An UNFURLING of a compact connected manifold M is a connected ∞-cyclic covering $\tilde{M}$ of M, or equivalently a principal Z-bundle over M. The quotient map $R \to T^1$ is a universal principal Z-bundle; hence there is a map $M \xrightarrow{f} T^1$ unique up to homotopy covered by a Z-equivariant map $\tilde{f} : \tilde{M} \to R$, under which the two ends $\pm\infty$ of R correspond to the two ends of $\tilde{M}$.

Furling (called gluing in $[S_2$, §5])†‡ reverses the passage from M to $\tilde{M}$ in some important special cases. Given a manifold N^n having two ends ε_- and ε_+ (specified in this order) we assume there exist arbitrarily small neighborhoods U_- and U_+ of these ends and homeomorphisms $\varphi_\pm : U_\pm \to N$, homotopic to $id|U_\pm$ fixing smaller neighborhoods of the ends. In case $U_- \cap U_+ = \phi$, we may glue together the ends of N by $\varphi_+^{-1}\varphi_-$ to obtain the FURLING

$$F(\varphi_-, \varphi_+) = N/\{x = \varphi_+^{-1}\varphi_-(x);\ x \in U_-\}\ .$$

This is a compact manifold.

For example, if $N = R^1$, let $U_- = (-\infty, -3)$, $U_+ = (+3, +\infty)$ and let $\varphi_\pm : U_\pm \to R$ be diffeomorphisms fixing neighborhoods of the ends $\pm\infty$ and translating a neighborhood of ± 4 so that $\varphi_\pm(\pm 4) = 0$. Then $F(\varphi_-, \varphi_+)$ is canonically diffeomorphic to $R^1/8Z = T^1$.

If there is a homeomorphism $h : N \to X$, then $h(U_-)$, $h(U_+)$, $h\varphi_- h^{-1}$, serve to furl x, and $F(\varphi_+, \varphi_-)$ will be canonically homeomorphic to $F(h\varphi_- h^{-1}, h\varphi_+ h^{-1})$. The latter is said to be the furling induced by h on X.

Typically we construct our φ_+, φ_- by engulfing when $n \geq 5$, or by splitting N into some $P^{n-1} \times R$ and using the furling of R above.

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†We avoid the full generality of the treatment in $[S_2$, §5].

‡Compare a roughly equivalent process in [K-S, Essay II, §1].

CAT UNIQUENESS THEOREM 3.1. [S_2, Theorem 5.2] *Any two furlings of N, say $F(\varphi_-, \varphi_+)$ and $F(\varphi_-^*, \varphi_+^*)$, are homeomorphic. Furthermore, the homeomorphism as constructed will be the identity on the common subset $N - (U_- \cup U_+ \cup U_-^* \cup U_+^*)$.*

The proof is a pleasant exercise that the reader should probably pause to do. Hint: In case $\varphi_- = \varphi_-$ the result is obvious since we can just throw away a neighborhood of ϵ_- in N containing all images of points where φ_+ and φ_+^* disagree. Similarly with roles of + and - interchanged. This suffices. □

The furling construction and the proof of uniqueness are so simple that a number of observations can be made with no further effort. However the reader will be better motivated if he postpones reading these observations until they are used in §5 and §6.

PAIRWISE UNIQUENESS THEOREM 3.2. *If N' is a submanifold of N closed in N, and throughout the entire discussion all the $\varphi_\pm$ mentioned respect N', then the quotient of N' in any furling is a compact submanifold, and the homeomorphisms of uniqueness $F(\varphi_-, \varphi_+) \to F(\varphi_-^*, \varphi_+^*)$ induce a homeomorphism of the quotients of N'.* □

COMPLEMENT 3.2.1. *If N' is CAT and all the $\varphi_\pm$ are CAT on N', then the quotients of N' are CAT and the homeomorphisms of uniqueness give CAT isomorphisms of the respective quotients of N'. (We do not exclude $N = N'$!)* □

COMPLEMENT 3.2.2. *If N has a CAT structure near N' and all the $\varphi_\pm$ are CAT near N', then each furling inherits a CAT structure near the quotient of N'; and the homeomorphisms of uniqueness are CAT imbeddings near the respective quotients of N'.* □

COMPLEMENT 3.2.3. *Suppose N has a structure of product with R specified near N', given by an open imbedding of a neighborhood of N' into a product $A \times R$ carrying N' onto a set $N_0' \times R$. And suppose all $\varphi_\pm$ are a product with $id|A$ near N'. Then near the quotient of N', any furling inherits a product structure given by an open imbedding into $A \times T^1$ carrying the quotient of N' onto $N' \times T^1$. Any homeomorphism of uniqueness will be a product with $(id|A)$ near the respective quotients of N'.* □

Any furling $F(\varphi_-, \varphi_+)$ has a preferred unfurling $\tilde{F}(\varphi_-, \varphi_+)$ namely the quotient of $Z \times N$ by identification of $(m,x) \in m \times U_+$ to $(m+1, \varphi_-^{-1}\varphi_+(x)) \in (m+1)\times U_-$, whenever $m \in Z$ and $x \in U_+$. The canonical covering translation $T : \tilde{F} \to \tilde{F}$ sends (m,x) to $(m+1,x)$.

In any case of interest to us, the unfurling $\tilde{F}(\varphi_-, \varphi_+)$ is isomorphic to any $m \times N$ by an engulfing process that sends positive end to positive end. (We shall not use this fact.)

It is clear from the construction of the uniqueness homeomorphism

$$\theta : F(\varphi_-, \varphi_+) \to F(\varphi_-^*, \varphi_+^*)$$

by reduction to special cases where either $\varphi_- = \varphi_-^*$ or $\varphi_+ = \varphi_+^*$, that θ is covered by a homeomorphism $\tilde{\theta} : \tilde{F}(\varphi_-, \varphi_+) \to \tilde{F}(\varphi_-^*, \varphi_+^*)$ commuting with the canonical covering translations.

Regarding these ∞-cyclic coverings as principal Z-bundles, and classifying by maps to $B_Z = T^1$ we obtain

PROPOSITION 3.3. *Every furling $F(\varphi_-, \varphi_+)$ has a preferred homotopy class of maps to T^1, and the homeomorphisms of uniqueness from 3.1 respect these preferred homotopy classes.* □

Let $\hat{N} \to N$ be a locally trivial bundle. Since $\varphi_\pm$ is homotopic to $id|U_\pm$ fixing a smaller neighborhood of $\varepsilon_\pm$, it is clear how the furling construction extends to produce a furled locally trivial bundle $\hat{F}(\hat{\varphi}_-, \hat{\varphi}_+)$ over $F(\varphi_-, \varphi_+)$. Also any isomorphism of uniqueness $\theta : F(\varphi_-, \varphi_+) \to F(\varphi_-^*, \varphi_+^*)$ is naturally covered by an isomorphism of bundles $\hat{\theta} : \hat{F}(\hat{\varphi}_-, \hat{\varphi}_+) \to \hat{F}(\hat{\varphi}_-^*, \hat{\varphi}_+^*)$. Applying this to locally trivial principal Z-bundles and classifying we get:

PROPOSITION 3.4. *If a preferred homotopy class of maps $N \to T^1$ is given there is a preferred homotopy class of maps $F(\varphi_-, \varphi_+) \to T^1$ coinciding up to homotopy with the first on $N - (U_- \cup U_+)$. The isomorphisms of uniqueness respect this preferred class.* □

4. REDUCTION OF THEOREM A TO A HANDLE LEMMA.

The existence part of the theorem follows easily from the relative uniqueness part of the theorem. This in turn follows by an induction over coordinate charts from the case where $M = R^m$. This case in turn follows by induction on handles in R^m from the HANDLEWISE VERSION OF UNIQUENESS, where $M = R^m = R^k \times R^n$ and the ambient isotopy h_t of $id|Q$ yields $h_1\nu' = \nu$ only near $C \cup B^k \times B^n$ in Q, and instead of being small, h_t is required to have compact support. These kinds of reductions are becoming standard, and we merely refer here to [K-S_2] where several are carried out, see [K-S_2, Essay I §4] for example.

We now deal with the HANDLEWISE VERSION. As M is contractible the microbundles ν and ν' admit trivializations $g : M \times R^2 \to Q$ and $g' : M \times R^2 \to Q$. Certainly we can choose g and g' so that $g'(M \times R^2) \subset g(M \times R^2)$ and $g'(y \times R^2) \subset g(y \times R^2)$ for all $y \in M$ near C. Using the theorems of Kister and Kneser mentioned in the introduction we can arrange further that for all y near C in M, $g'(y \times R^2) = g(y \times R^2)$ and $g^{-1}g' \mid y \times R^2$ is in the orthogonal group $O(2)$. Using a DIFF approximation of the resulting map of a neighborhood of C to $O(2)$, we furthermore arrange that the imbedding

$$g^{-1}g' : R^m \times R^2 \to R^m \times R^2$$

is a DIFF imbedding near $C \times R^2$. Note now that if $h = g^{-1}g'$ were a DIFF imbedding near $R^m \times 0$ we could get our result from the relative uniqueness theorem for DIFF tubular neighborhoods.† It remains then to arrange this by dint of an admissible isotopy according to the following.

CODIMENSION 2 HANDLE LEMMA 4.1. *Consider a pairwise open imbedding* $h : R^k \times (R^{n+2}, R^n) \to (Q^{m+2}, M^m)$, $m = k + n$, *to a DIFF pair, so that* h *is a DIFF imbedding on* $R^m = R^k \times R^n$ *and near* $C = (R^k - \mathring{B}^k) \times R^{n+2}$.

Provided that $m + 2 \neq 4$, *there exists a pairwise isotopy* h_t, $0 \leq t \leq s$, *of* $h = h_0$ *so that*

1) the isotopy fixes h *on* R^m, *near* C *and outside a compact set.*

2) h_1 *is a DIFF imbedding on a neighborhood of* R^m.

5. THE TOWER DIAGRAM FOR SOLVING THE HANDLE PROBLEM.

To prove the handle lemma 4.1, we construct the following Diagram 5-a.

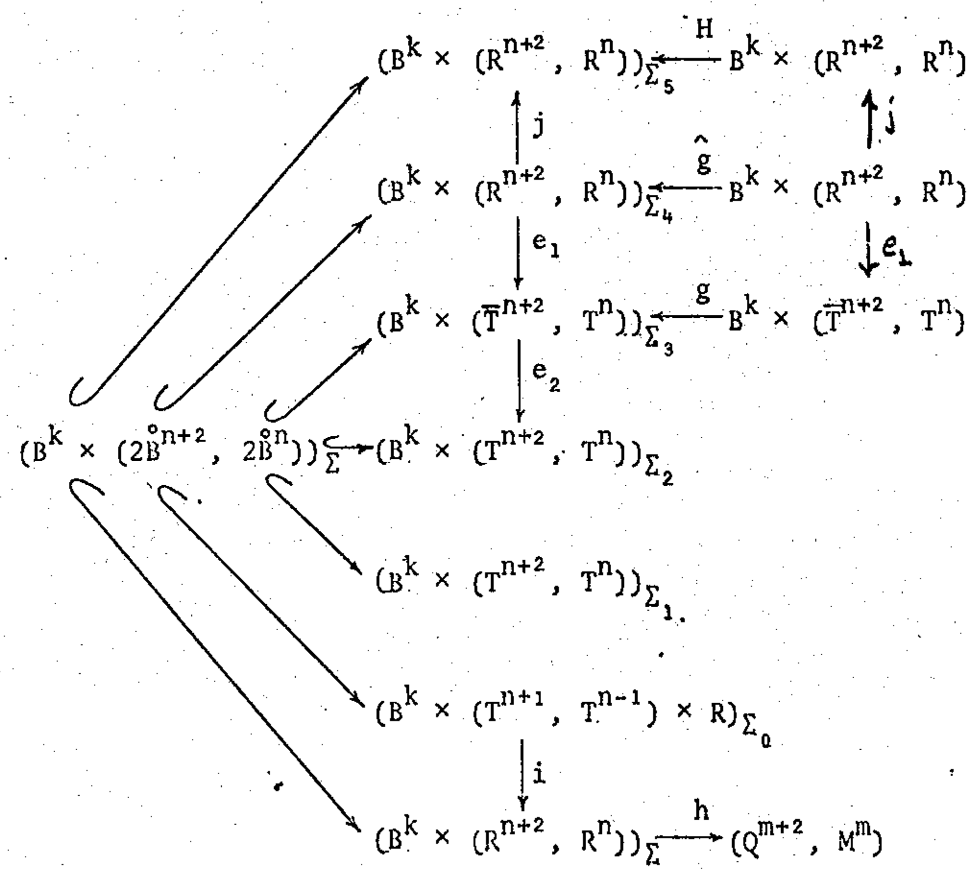

DIAGRAM 5-a

It has had many precursors, particularly [K-S_2, Essay I, § 3]; so of those parts that are familiar we shall be content with a brief description.

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†This is quickly done by hand as follows. Let r_t be a DIFF homotopy of $\mathrm{id}|R^2$ fixing $R^2 - B^2$ and 0 to a map r mapping $\frac{1}{2}B^2$ to 0. For small $\varepsilon > 0$ consider the DIFF homotopy $h_t(x,y) = (h_t'(x,y), h''(x,y))$, $(x,y) \in R^m \times R^2$, which deforms only the R^m component h' of h, by the formula $h_t'(x,y) = h'(x, \varepsilon r_t(y/\varepsilon))$. One verifies that if ε is sufficiently small h_t remains nonsingular and arbitrarily C° close to h; then h_t is necessarily a diffeotopy with compact support in $\mathring{B} \times R^2$ and fixing $R^m \times 0$; it arrives at a DIFF embedding h_1 which, near $R^m \times 0$, respects projection to R^m.

Σ is the DIFF structure pulled back by h, so that in the diagram h is a DIFF isomorphism. This Σ is standard near the boundary ∂ and on $B^k \times R^n$. By restriction we get the DIFF structure Σ on $B^k \times (2\mathring{B}^{n+2}, 2\mathring{B}^n)$.

The structures $\Sigma_0, \Sigma_1, \ldots, \Sigma_5$ are to be chosen standard near the boundary and such that the commutative triangles of canonical injections on the left will all be DIFF imbeddings near $B^k \times 2\mathring{B}^n$. Bear this in mind even when it is not explicitly stated.

By induction on n one forms a DIFF imbedding à la Novikov

$$(T^{n+1}, T^{n-1}) \times R \to (R^{n+2}, R^n)$$

that is the identity on $[-2,2]^{n+2}$. Producting with $id|B^k$ we get the imbedding i in the diagram. The structure Σ_0 is the pull-back of Σ by i.

The first new device is the construction of Σ_1 from Σ_0.

(a) One forms a DIFF pairwise furling of $(B^k \times (T^{n+1}, T^{n-1}) \times R)_{\Sigma_0}$ using DIFF engulfing respecting $B^k \times T^{n-1} \times R$. This requires $m + 2 \geq 5$.† The neighborhoods U_- and U_+ involved are chosen disjoint from $B^k \times 2B^{n+2}$.

(b) One regards $B^k \times (T^{n+2}, T^n)$ as the furling of $B^k \times (T^{n+1}, T^{n-1}) \times R$ derived from the standard furling above of R yielding T^1.

(c) Then one applies the uniqueness result for furlings that respect $B^k \times T^{n-1} \times R$, are DIFF on it, and simultaneously are DIFF near the boundary (for the standard structures). This provides a pairwise homeomorphism of the first furling onto the second, thereby endowing the second with a DIFF structure $(B^k \times (T^{n+2}, T^n))_{\Sigma_1}$ that is standard on $B^k \times T^n$ and near the boundary, and equals Σ near $B^k \times 2B^n$.

An auxiliary structure Σ^* is formed from Σ_1 by unfurling the first circle factor complementary to T^n, getting $(B^k \times T^{n+1} \times R)_\Theta$ say, then furling again by DIFF engulfing using neighborhoods of $\pm\infty$ disjoint from $B^k \times T^{n+1} \times 2B^1$. Near the boundary this furling process is to coincide with that from the standard furling of R. This construction is a little simpler than the construction of Σ_1 from Σ_0 as it is not pairwise; the submanifold $B^k \times T^n$ is never touched since the circle T in question is complementary to T^n rather than part of it. Thus we get $(B^k \times (T^{n+2}, T^n))_{\Sigma^*}$ standard near the boundary and equal Σ_1 on $B^k \times T^{n+1} \times 2\mathring{B}^1$.

By precisely the same procedure of unfurling and refurling applied to Σ^*, this time along the second T^1 factor complementary to T^n, one derives the structure Σ_2 from Σ^*.

Let $\overline{T}^2$ be the standard finite covering $R^2/s8Z^2$ of $T^2 = R^2/8Z^2$ for some sufficiently large positive integer s to be determined later (in §6). Let

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†The engulfing diffeomorphisms are built up from two sorts: those obtained by the engulfing theorem and having support in the complement of $B^k \times T^{n-1} \times R$; and those that slide a smooth tubular neighborhood of $B^k \times T^{n-1} \times R$ over itself in a standard fashion.

$\bar{e}_2 : \bar{T}^2 \to T^2$ be the quotient map and let $e_2 : B^k \times \bar{T}^{n+2} \to B^k \times T^{n+2}$ be the product of $\mathrm{id} \mid (B^k \times T^n)$ with $\bar{e}_2$.

The pairwise diffeomorphism g will be homotopic to the identity, and equal to the identity near the boundary, and on T^n. As one might expect g is tricky to obtain, and we postpone this "torus problem" to a section of its own, §6.

e_1, Σ_4 and $\hat{g}$ arise by passage to the standard universal covering of the diffeomorphism g. As $g \simeq \mathrm{id}$, the diffeomorphism $\hat{g}$ is BOUNDED; i.e., $\sup\{|g(\vec{x}) - \vec{x}|;\ \vec{x} \in B^k \times R^{n+2}\} < \infty$.

Let $J : R^{k+n+2} \to 5\mathring{B}^{k+n+2}$ be a radial homeomorphism that is the identity on $4B^{k+n+2} \supset B^k \times 2B^{n+2}$, and let j be the restriction of J. Since g is bounded and $\hat{g} = \mathrm{id}$ on the boundary, it follows that jgj^{-1} can be extended by the identity to a homeomorphism G of all of R^{k+n+2}; H is $G \mid B^k \times R^{n+2}$. Σ_5 is chosen so that H is a diffeomorphism. Note that $\Sigma_5 = \Sigma$ near $B^k \times 2B^{n+2}$.

The Alexander isotopy G_t of G to the identity, namely $G_t(\vec{x}) = \vec{x}$ if $t = 0$ and $G_t(\vec{x}) = tG(\vec{x}/t)$ if $t \in (0,1]$, restricts to an isotopy $H_t = G_t \mid (B^k \times R^{n+2})$ of H to $\mathrm{id} \mid B^k \times R^{n+2}$. It has compact support in $\mathring{B}^k \times R^{n+2}$ and fixes $B^k \times R^n$ pointwise.

We define $h_t = hH_t$. Then h_t obviously satisfies 1), and $h_1 = hH_1$ is a DIFF embedding near $B^k \times B^n$ because H_1 fixes $B^k \times R^n$ and $\Sigma_5 = \Sigma$ near $B^k \times 2\mathring{B}^n$.

6. Solution of the Torus Problem.

Given Σ_1 and Σ_2 we seek some standard finite covering Σ_3 of Σ_2 and a diffeomorphism

$$g : B^k \times (\bar{T}^{n+2}, T^n) \to (B^k \times (\bar{T}^{n+2}, T^n))_{\Sigma_3}$$

equal to the identity on $B^k \times T^n$ and near the boundary, and homotopic to the identity.

Initially we exclude the cases $m = 3$ and $m = 4$. These in fact follow from the case $m \geq 5$ as we will explain at the very last.

Assertion 1. *For $m \neq 3, 4$, there exists a diffeomorphism*

$$f : V^m \times T^2 \to (B^k \times T^{n+2})_{\Sigma_2}$$

that is a product with T^2 near the boundary and admits a lifting

$$\tilde{f} : V^m \times R^2 \to (B^k \times T^n \times R^2)_{\tilde{\Sigma}_2}$$

that is a product with R^2 near the boundary. Here $\tilde{\Sigma}_2$ is the natural covering of Σ_2.

Proof of Assertion 1. We retrace (but do not redo) the construction of Σ_2 from Σ_1 via Σ^*.

First split by 2.1 to obtain a diffeomorphism $W^{m+1} \times R \to (B^k \times T^{n+1} \times R)_\Theta$ that is a product with R near the boundary. Recalling that $W^{m+1} \times T^1$ is the standard furling of $W^{m+1} \times R$, we apply the uniqueness theorem for furlings to obtain a diffeomorphism

$$f^* : W^{m+1} \times T^1 \to (B^k \times T^{n+2})_{\Sigma^*}$$

that is a product with $\mathrm{id} \mid T^1$ near the boundary by Complement 3.2.3. (This complement initially provides an f^* that is merely a product with some orientation preserving diffeomorphism $T^1 \to T^1$; but an isotopy quickly makes it $\mathrm{id}|T^1$.)

Unfurling source and target of f^* along the other T^1 factor complementary to T^n and splitting again, we get a diffeomorphism

$$V^m \times R \times T^1 \overset{\simeq}{\to} (B^k \times T^n \times R \times T^1)_{\Theta^*}$$

that is a product with $R \times T^1$ near the boundary.

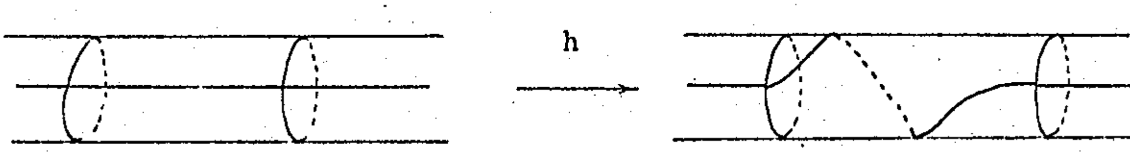
Thus applying the uniqueness theorem once more we get a diffeomorphism

$$f : V^m \times T^2 \to (B^k \times T^{n+2})_{\Sigma_2}$$

that is a product with T^2 near the boundary, by Complement 3.2.3 again.

The map f lifts to a diffeomorphism $\tilde{f}$ with $q\tilde{f} = fq$, where q simply arises from the quotient map $R^2 \to T^2$, if and only if the induced map f_* of fundamental groups respects the projections to $\pi_1(T^2) = Z^2$. This is dictated by the behavior of f on the boundary if it is nonempty; i.e., if $k \geq 1$. (What is more f_* is then evidently a product with $\mathrm{id}|(\pi_1(T^2))$.) If $k = 0$, one can still see easily enough that f_* respects the projections to the last two factors $\pi_1(T^1)$ by using propositions 3.3 and 3.4, and retracing the construction of f just given. This means that f_* respects the projections to $\pi_1(T^2)$ as required. Here is the one spot 3.3 and 3.4 are used.

The argument here is subtle. As an aid in understanding it, consider $S^1 \times R$ and a homeomorphism h which gives one end a full twist, as pictured



The reader should convince himself that the two furlings, one standard and the other induced from the standard furling by h, are canonically homeomorphic by a homeomorphism which does *not* have a twist in it.

If $k \geq 1$, we select that lift $\tilde{f}$ of f which is a product with $\mathrm{id}|R^2$ near a base-point b of the boundary. Since f is a product with $\mathrm{id}|T^2$ near the boundary, this is possible and $\tilde{f}$ will forcibly be a product with $\mathrm{id}|R^2$ near the connected component of the boundary containing b. This is the whole boundary if $k \geq 2$. If $k = 1$, then, near the other component of boundary, $\tilde{f}$ is a product with translation in R^2 by an element of $8Z^2$; but it is clear how to rechoose f altering it near the second boundary component so that the identity translation appears.

This completes the proof of Assertion 1 in all cases. □

ASSERTION 2. *There exists a DIFF automorphism ψ of $(B^k \times T^n \times R^2)^{\sim}_{\Sigma_2}$ fixing points outside some compactum $|\psi|$ in $\mathring{B}^k \times T^n \times R^2$, such that $\psi\, \tilde{f}(V^m \times 0) = B^k \times T^n \times 0$.*

PROOF OF ASSERTION 2. To simplify notation, from this point we will suppose $k = 0$; i.e., $m = n$. The adjustments for the general case are easily summarized at the end of the proof.

Choose a constant λ so large the tubular neighorhood $N_V = \tilde{f}(V^m \times \lambda B^2)$ of $\tilde{f}(V^m)$ contains $T^m \times 0$. Then choose a smooth tubular neighborhood N_T of $T^m \times 0$ in $(T^m \times R^2)^{\sim}_{\Sigma_3}$ so small that $N_T \subset \mathring{N}_V$. See Figure 6-a.

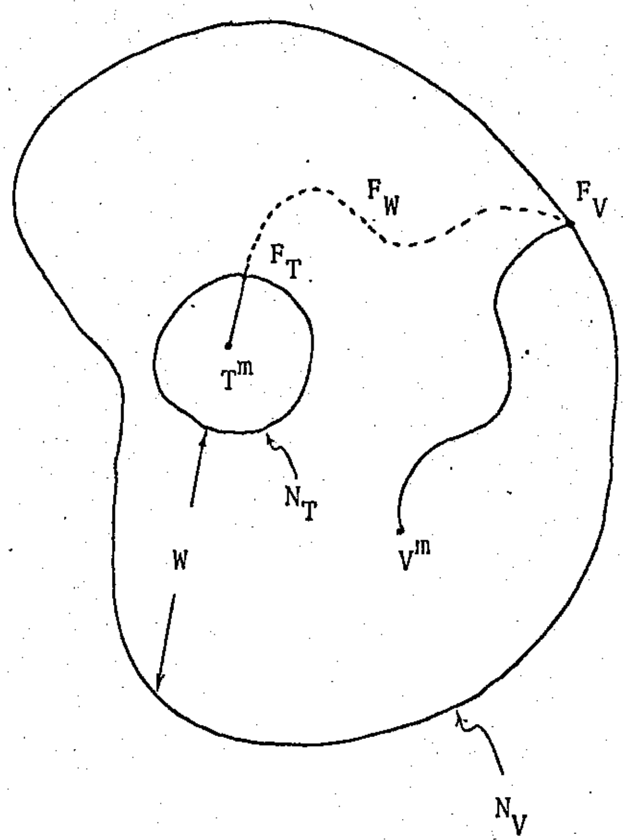


Figure 6-a

Now $f \mid (y \times T^2)$ followed by projection to T^2 is homotopic to the identity for each point $y \in V^m$. Therefore $\bar{f} \mid (y \times R^2)$ followed by projection to R^2 is degree 1, and it follows that the projection to $R^2 - 0$ restricted to the circle bundle $\dot{N}_V$ is degree 1 on each fiber. Since the fibers of $\dot{N}_T$ link T^m simply, one can also verify that projection to $R^2 - 0$ is of degree 1 on each fiber of $\dot{N}_T$. Hence, in each case the projection provides a homotopy trivialization of the circle bundle. Now $G(2)/O(2)$ is contractible where $G(2)$ is the space of degree ± 1 homotopy equivalences of the circle. As $G(2)/O(2)$ classifies homotopy trivialized circle bundles we conclude that the projections of $\dot{N}_V$ and $\dot{N}_T$ to $R^2 - 0$ can be deformed to smooth orthogonal circle bundle trivializations; in particular to smooth fibrations over the circle $S^1 \subset R^2 - 0$.

The cobordism $W = N_V - \mathring{N}_T$ is easily seen to be an h-cobordism; we leave this as an exercise.

Thus the projection $W \to (R^2 - 0)$ once deformed as above so that $\partial W \to (R^2 - 0)$ is a fibration over S^1, is susceptible to Farrell's fibering theorem.

Applying it, we get a further deformation of the projection $W \to (R^2 - 0)$ to become a fibration of W over S^1. Any fiber F_W necessarily provides an h-cobordism from a fiber F_T of $\dot{N}_T \to S^1$ to a fiber F_V of $\dot{N}_V \to S^1$. The s-cobordism theorem says that F is differentiably a product cobordism.

We now have smoothly imbedded product cobordisms in $(T^m \times R^2)_{\overline{\Sigma}_3}$ running $\overline{f}(V^m) \cong F_V \cong F_T \cong T^m$, as indicated in Figure 6-a. These quickly yield three automorphisms whose composition ψ completes the proof of Assertion 2 for $k = 0$. □

In case $k > 0$, the above argument produces an automorphism φ provided all constructions near the boundary are kept standard. Although none of the three automorphisms composing ψ are the identity near the boundary, their composition is easily made to fix the boundary. □

CONSTRUCTION OF g (FOR $m \neq 3, 4$) Choose the integer s determining the covering $e_2 : \overline{T}^2 \to T^2$ so that the compactum $|\psi|$ in $(B^k \times T^n \times R^2)_{\tilde{\Sigma}_2}$ is entirely in a fundamental domain $B^k \times T^n \times (-4s, 4s]^2$. Then ψ determines a unique $8sZ^2$ - - equivariant automorphism $\tilde{\psi}$ of $(B^k \times T^n \times R^2)_{\tilde{\Sigma}_2}$, that coincides with] on this fundamental domain.

This $\tilde{\psi}$ covers an automorphism $\overline{\psi}$ of $(B^k \times T^n \times \overline{T}^2)_{\Sigma_3}$ such that if $\overline{f} : V^m \times 1 \times \overline{T}^2 \to B^k \times T^m \times \overline{T}^2$ covers f for e_2, then $\overline{\psi f}$ maps $V^m \times 0$ onto $B^k \times T^n \times 0$. We use this to identify V^m to $B^k \times T^n$ thus producing the diffeomorphism

$$\overline{\psi f} : B^k \times \overline{T}^{n+2} \to (B^k \times \overline{T}^{n+2})_{\Sigma}$$

equal to the identity near the boundary and on $B^k \times T^n \times 0$.

If $k \geq 1$, we set $g = \overline{\psi f}$. This g is homotopic to the identity as it fixes the boundary.

If $k = 0$, it is not clear that $\overline{\psi f}$ is homotopic to the identity (equivalently fixes π_1). It does fix $\pi_1 T^n$ and does respect projection to $\pi_1 \overline{T}^2$. The inverse of the matrix of the π_1-map induced by $\overline{\psi f}$ is a linear automorphism ω of $\overline{T}^{n+2}$ (here identified canonically to T^{n+2}), that fixes T^n and respects projection to $\overline{T}^2$. We can now define

$$g = \overline{\psi}\, \overline{f}\, \omega$$

completing the construction of g in the last case $k = 0$. □

CONSTRUCTION OF g FOR $m = 3$ AND 4. The construction for $m \geq 5$ of Σ_2, Σ_3, and g, starting from the structure Σ_1 applies to <u>any</u> structure

$$(B^k \times (T^{n+2}, T^n))_{\Sigma_1}$$

that is standard on $B^k \times T^n$ and standard near the boundary. It makes no difference where Σ_1 comes from.

If $m \equiv k + n = 3$ or 4 we can apply it to $\Sigma_1 \times T^2$ with m and n each increased by 2. The two structures produced may as well be the structures $\Sigma_2 \times T^2$ and $\Sigma_3 \times T^2$. (Recall that Σ_2 and Σ_3 *were* defined even for $m = 3$ and 4.) We thereby obtain a diffeomorphism

$$g'' : B^k \times (\overline{T}^{n+2}, T^n) \times T^2 \to (B^k \times (\overline{T}^{n+2}, T^n))_{\Sigma_3} \times T^2$$

that is the identity on $B^k \times T^n \times T^2$ and near the boundary. We unfurl this along the last circle factor and apply the (relative and pairwise) s-cobordism theorem in dimension $m + 4$ to obtain a pairwise diffeomorphism

$$g' : B^k \times (\overline{T}^{n+2}, T^n) \times T^1 \to B^k \times (\overline{T}^{n+2}, T^n)_{\Sigma_3} \times T^1$$

that is again the identity on $B^k \times T^n \times T^1$ and near the boundary. Repeating this process once more yields the diffeomorphism g required. □

The proof of the handle lemma 4.1, and of our principal result Theorem A is now complete. □

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