Prime knots and concordance

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Prime knots and concordance

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This paper proves that any knot is concordant to a prime knot; it thus solves Problem 13 of (3). In doing so it makes an exploration of a fairly general method of proving that a knot is a prime. Throughout, the word 'knot' means a knot of S^1 in S^3 (orientations being here irrelevant); occasionally reference will be made to the idea of a knotted arc spanning a 3-ball.

The method used in this paper is to take the connected sum of two knots and find a concordance to what is intuitively a more complicated knot which can be shown to be prime. Thus, there are relations between concordance classes of prime knots of the form $A + B = C$. One might wonder if one can determine an independent set of generators for the group of concordance classes. Litherland (4) has proved that the classes of the torus knots do form an independent set, though it does not generate.

The techniques used here are the usual elementary methods of 3-manifold topology. The paper should be interpreted as being in the piecewise linear category, but that is not very important. Standard definitions and results of knot theory may be found in (5). The word 'tangle' is borrowed from (1) and is used to mean a finite set of disjoint arcs properly embedded in a 3-ball.

Definition. A tangle t in a 3-ball B is *prime* if it has the following properties:

(1) Any 2-sphere in B, which meets t transversely in two points, bounds in B a ball meeting t in an unknotted spanning arc.

(2) The arcs of t cannot be separated by a disc properly embedded in B.

LEMMA 1. *Let k be a prime non-trivial knot in S^3. Then there exists, embedded in S^3, a 2-sphere meeting k transversely in four points and separating S^3 into 3-balls A and B such that*

(i) $(A, A \cap k)$ *is a trivial tangle (i.e. $A \cap k$ consists of two unlinked unknotted arcs spanning A);*

(ii) $(B, B \cap k)$ *is a prime tangle.*

Proof. Let X and Y be (small) disjoint 3-balls in S^3 each of which meets k in a single unknotted spanning arc. Let Z be a regular neighbourhood in $\overline{S^3 - (X \cup Y)}$ of an arc α that joins a point of ∂X to one in ∂Y but is otherwise disjoint from $X \cup Y \cup k$. This Z is required to meet ∂X and ∂Y each in a disc, to be disjoint from k, and to be such that the

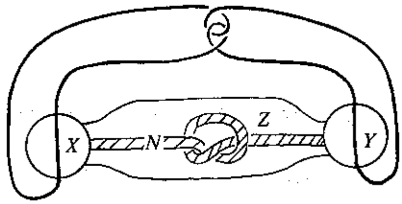

Fig. 1

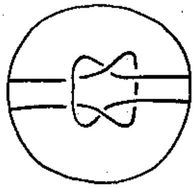

Fig. 2

3-ball $X \cup Z \cup Y$ has the property (*) that there exists in its boundary a loop, essential in $S^3 - k$, but linking k with zero linking number. (Existence of such a Z follows immediately by taking m_x and m_y to be meridians of k contained in ∂X and ∂Y and α to be an arc so that $\alpha m_x \alpha^{-1} m_y$ is the composite of two distinct elements in a Wirtinger presentation of the knot-group.) Let N be a regular neighbourhood in Z, meeting ∂Z regularly, of a knotted spanning arc in Z, joining a point of ∂X to a point of ∂Y. Let A be

$$X \cup N \cup Y$$

and let B be $\overline{S^3 - A}$.

Now $(A, A \cap k)$ is trivial from the construction. To establish (1), note first that the ball A inherits property (*) from $X \cup Z \cup Y$. If F is a 2-sphere in B meeting k in two points, suppose that F bounds a ball β in B and $(\beta, \beta \cap k)$ is knotted. Then, because k is prime, $(\overline{S^3 - \beta, k - \beta})$ is an unknotted ball pair. But $A \subset \overline{S^3 - \beta}$, so any loop in $A - k$ with zero linking number with k is inessential. This contradicts (*) and so $(\beta, \beta \cap k)$ is unknotted.

Lastly suppose that D is a disc properly embedded in B, separating B into two balls, each containing one of the arcs of $B \cap k$. Assume that D is transverse to the annulus $B \cap \partial Z$. Suppose γ is any curve encircling that annulus (i.e. $[\gamma]$ is a generator of $H_1(B \cap \partial Z)$). If γ is in one of the components of $B - D$, then γ cannot link the spanning arc of $B \cap k$ contained in the other component. But, in B, γ has unit linking number with each arc of $B \cap k$, so $\gamma \cap D \neq \phi$ for any such γ. Hence D contains arcs running from one boundary component of the annulus $(B \cap \partial Z)$ to the other. Consider such an arc extreme-most on D. Such an arc is contained in the boundary of a subdisc δ of D, the remainder of whose boundary is an arc in ∂B. However N is a neighbourhood of a *knotted* arc spanning Z so that the boundary of δ would have to be knotted in S^3. This contradiction establishes the lemma.

Note that the lemma also holds for the trivial knot, the proof being amended by letting α be an arbitrary arc from X to Y and noting that (1) is trivial in this case.

The tangle shown in Fig. 2 will be called a *clasp*.

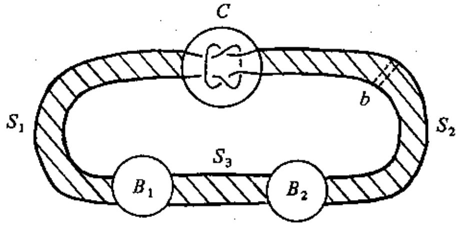

Fig. 3

LEMMA 2. *A clasp is a prime tangle.*

Proof. As each spanning arc of the clasp is unknotted, (1) is immediate. If there were a properly embedded 2-disc separating the two arcs of the clasp then the clasp would be the trivial tangle. However, a trivial tangle can be added to the clasp (on the outside of the ball in Fig. 2) to create the square knot (or reef knot). The square knot would thus be a 2-bridge (rational) knot; this is known to be false (see for example Theorem 2 of (2) or 4D15 of (5)).

THEOREM. *Any knot is concordant to a prime knot.*

Proof. Any knot is expressible as a sum of prime knots, and knot addition is compatible with concordance, so it is sufficient to show that the sum of just two prime knots is concordant to a prime knot. Let k_1 and k_2 be prime knots in S^3. Now, Lemma 1 provides a method of placing a prime knot k in S^3 expressed as the union of two 3-balls A and B. Because $(A, A \cap k)$ is the trivial tangle, $A \cap k$ can be isotoped into $\partial A = \partial B$. Then, k is entirely in B; it is the union of two arcs in ∂B and two spanning arcs with properties (1) and (2). Let B_1, B_2 and C be disjoint 3-balls in S^3 connected by strips (copies of $I \times I$) S_1, S_2 and S_3 as shown in Fig. 3. Let k_1 be embedded in B_1, and k_2 in B_2, in the way described above, so that the arcs of the k_i in ∂B_i coincide with the appropriate arcs at the ends of the strips. Let two arcs forming a clasp be embedded in C as shown in Fig. 3.

The union of the six spanning arcs in B_1, B_2 and C, and the six relevant arcs of the boundary of the strips $\left(\text{i.e. } \left(\partial \bigcup_{i=1}^{3} S_i\right) - (B_1 \cup B_2 \cup C)\right)$ form a new knot $\hat{k}$. This $\hat{k}$ is concordant to $k_1 \# k_2$ by performing a band-move along the band b as shown. That move immediately changes $\hat{k}$ to $k_1 \# k_2$ together with an unknot that does not link it.

It remains to prove that the knot $\hat{k}$ is a prime knot. Suppose that F is a 2-sphere embedded in S^3 cutting $\hat{k}$ in two points, splitting $\hat{k}$ into two non-trivial knotted ball-pairs. Suppose F is transverse to $\partial(B_1 \cup B_2 \cup C)$. Let γ be a simple closed curve of this intersection, innermost on F, and let D be the innermost disc in F that it bounds. Because $F \cap \hat{k}$ consists of only two points, γ can be chosen so that $D \cap \hat{k}$ consists of one point or none at all.

(*a*) If $D \cap \hat{k}$ is empty proceed as follows: If $D \subset B_1 \cup B_2 \cup C$, then D splits one of those balls into two components (each a ball) one of which is (by property (2)) disjoint from $\hat{k}$. There is thus an isotopy, supported on a neighbourhood of that component, which keeps $\hat{k}$ fixed and moves D out of the ball in question. This reduces the number of components of $F \cap \partial(B_1 \cup B_2 \cup C)$. A similar argument covers the case when

$$\mathring{D} \subset S^3 - (B_1 \cup B_2 \cup C).$$

(*b*) Suppose $D \cap \hat{k}$ is one point: If $D \subset B_1 \cup B_2 \cup C$, suppose $D \subset B_1$ (the other cases are exactly the same). Then γ separates ∂B_1 into two discs one of which, Δ, meets $\hat{k}$ in one point. By property (1), $D \cup \Delta$ bounds in B_1 a 3-ball meeting $\hat{k}$ in an *unknotted* spanning arc. Hence D may be isotoped across this ball (to just the other side of Δ) reducing the number of components of $F \cap \partial(B_1 \cup B_2 \cup C)$ without changing the types of the knots into which F separates $\hat{k}$. (Note that the number of components of $F \cap \partial\ (B_1 \cup B_2 \cup C)$ may be reduced by more than one by such a manoeuvre.) If $\mathring{D} \subset S^3 - (B_1 \cup B_2 \cup C)$ a similar argument applies.

Iteration of the above technique leads to a situation where $F \cap \partial(B_1 \cup B_2 \cup C) = \phi$. If $F \subset (B_1 \cup B_2 \cup C)$ there is by property (1) an immediate contradiction to the supposition that F provides a non-trivial factorization of $\hat{k}$. If $F \subset S^3 - (B_1 \cup B_2 \cup C)$, assume that F is transverse to $\bigcup_{i=1}^{3} S_i$ and that (by the usual innermost curve argument) F has been isotoped to remove closed curves of $F \cap \bigcup_{i=1}^{3} S_i$. Thus it can be assumed that $F \cap \bigcup_{i=1}^{3} S_i$ is one single arc α spanning S_j. If the end points of α belong to the same component of $\partial S_j \cap \hat{k}$, then F factorises $\hat{k}$ trivially; this is not so. Thus one of the end points of α belongs to each of the components of $\partial S_j \cap \hat{k}$. If l is a loop circling once around the set $(B_1 \cup B_2 \cup C) \cup \bigcup_{i=1}^{3} S_i$ (i.e. a spine of that set), l can be chosen to meet α in just one point. This means that the loop l has non-zero intersection number with the 2-sphere F in S^3; this is never true.

This concludes the proof of the theorem itself. It may well be that the method used, for proving the knot $\hat{k}$ is prime, is more interesting than the theorem itself. This $\hat{k}$ was constructed by inserting tangles, with two strings, that satisfy properties (1) and (2), into balls banded together by strips into a circuit as in Figure 3. Similarly, prime knots can be created by taking n such tangles $(n \geqslant 2)$ and inserting them into n balls banded together in any way into a circuit. In fact, the pattern of strips and balls may be taken to correspond to any graph in S^3, with a ball at each vertex and a strip along each edge, provided each vertex belongs to at least two edges and each edge has two distinct vertices. In this case tangles with r strings must be used at vertices where r edges meet, each tangle must be prime.

In practice the method of Lemma 1 may be unnecessarily complicated, for it is sometimes easy to see how to place a given knot in a 3-ball with two arcs on the boundary and a tangle in the interior with the two required properties. At the worst, Lemma 1 can be regarded as producing a presentation of a knot in a 3-ball with twelve more crossings than the minimal number (and this is only necessary if the knot is a 2-bridge knot). Thus if a knot k has prime factors k_i, $i = 1, 2, \ldots, n$, each with minimal crossing number r_i, then k is concordant to a knot with a presentation with $6+12n+\Sigma r_i$ crossings.

It is interesting to note that in his search recorded in (1) for concordances amongst knots with low numbers of crossings, Conway discovered one and only one example of a composite knot being concordant to a prime knot; namely, the granny knot is concordant to 10_{92} (in Tate's notation). In Conway's terminology this asserts that $3 \# 3$ is concordant to $.2.2.2.20$, the latter being the same as $(21, \overline{2}\overline{1})$ $(3, \overline{2}\overline{1})$ in 'algebraic' formulation. The tangle $(21, \overline{2}\overline{1})$ is the clasp of Fig. 2, and the tangle $(3, \overline{2}\overline{1})$, with two standard arcs added to make it a knot, is the granny placed in a ball so as to be a prime tangle. Thus this most simple example discovered by Conway is a direct illustration of the technique of this paper, using the graph consisting of a circuit with two vertices and two edges.

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