THE UNION OF FLAT $(n-1)$-BALLS IS FLAT IN R^n

BY ROBION C. KIRBY[1]

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THEOREM.[2] *Let β_1^{n-1} and β_2^{n-1} be two locally flat $(n-1)$-balls in R^n with $\beta_1 \cap \beta_2 = \partial\beta_1 \cap \partial\beta_2 = \beta^{n-2}$, where β^{n-2} is an $(n-2)$-ball which is locally flat in $\partial\beta_1$ and $\partial\beta_2$. Then $\beta_1 \cup \beta_2$ is a flat $(n-1)$-ball in R^n.*

This result has been announced by Černavskiĭ [1], but only for $n \geqq 5$ since his outlined proof uses engulfing. Our proof avoids engulfing and works for all n; a thorough knowledge of Cantrell and Lacher's version (see [2, §§4 and 5]) of Černavskiĭ's theorem is necessary to understand our proof.

We also have another proof of the following corollary which appears in [4].

COROLLARY. *Let $g: M^{n-1} \to N^n$ be an imbedding of an $(n-1)$-manifold into an n-manifold which is locally flat except on a set E. If $n > 3$, then E contains no isolated points (see [3] for the same result when M and N are spheres).*

PROOF. Let C be a neighborhood of an isolated point p in M which is homeomorphic to an $(n-1)$-ball, with g locally flat on $C - p$. Then split C into $(n-1)$-balls C_1 and C_2 so that $C = C_1 \cup C_2$ and $C_1 \cap C_2$ is an $(n-2)$-ball containing p. g is locally flat on C_1 and C_2 except at the point p on their boundaries. Then, since $n > 3$, g is flat on all of C_1 and C_2 by [5]. It follows from the theorem that $C_1 \cup C_2 = C$ is flat, so E has no isolated points.

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[2] *Added in proof.* Černavskiĭ has independently proven this theorem by similar methods.

Let R^n be Euclidean n-space, B^n be the unit n-ball, and R^k be imbedded in R^n as $R^k = \{x \in R^n | x_{k+1} = \cdots = x_n = 0\}$. We will coordinatize R^n by using $R^n = R^{n-2} \times R^2$ with polar coordinates on R^2. Thus points of R^n will be triples (z, r, θ) with $z \in R^{n-2}$, $r \geq 0$, and $\theta \in R$ and with the convention that $(0, r, 0)$ is a point on the positive x_{n-1}-axis and $(0, r, \pi/2)$ is a point on the positive x_n-axis. Let $H_\phi = \{(z, r, \theta) \in R^n | \theta = \phi\}$ and $D_\phi = H_\phi \cap B^n$. Note that $D_\pi \cup D_0 = B^{n-1}$ and $D_\pi \cap D_0 = B^{n-2}$. Let $W(\theta_1, \theta_2)$ be the wedge $\{(z, r, \theta) | \theta_1 \leq \theta \leq \theta_2\}$ and $\tilde{W}(\theta_1, \theta_2) = W(\theta_1, \theta_2) \cap B^n$.

PROOF OF THEOREM. Suppose β_1 and β_2 are given by imbeddings $f_1: D_\pi \to R^n$ and $f_2: D_0 \to R^n$. Since β^{n-2} is locally flat in $\partial \beta_1$ and $\partial \beta_2$, the closures of $\partial \beta_1 - \beta^{n-2}$ and $\partial \beta_2 - \beta^{n-2}$ are homeomorphic to $(n-1)$-balls. Then we may assume that $f_1(D_\pi) \cap f_2(D_0) = f_1(B^{n-2}) = f_2(B^{n-2}) = \beta^{n-2}$.

Since locally flat imbeddings of balls are flat, f_1 and f_2 extend to imbeddings of R^n into R^n (still called f_1 and f_2). We can require that the extensions are chosen so that $f_1(H_\pi) \cap f_2(D_0) = \beta^{n-2}$ and $f_2(B^n) \subset f_1(R^n)$. Then it suffices to show that $D_\pi \cup f_1^{-1} f_2(D_0)$ is locally flat. Let $f = f_1^{-1} f_2$.

Since $f(D_0) \cap H_\pi = B^{n-2}$, we can assume that $f(D_0) \subset W(0, \pi/4)$ by rotating $f(D_0)$ around R^{n-2} and away from H_π while fixing H_π. Then, in the coordinates of $f(B^n)$, we can rotate $f(D_\pi)$ close to $f(D_0)$, so we may as well assume that $f(D_\pi) \subset W(0, \pi/4)$ and lies between $H_{\pi/4}$ and $f(D_0)$ (see Figure 1).

Let $\tilde{h}: R^n - \text{int } H_0 \to R^n - \text{int } W(0, \pi/2)$ be the obvious homeomorphism which takes the wedge $W(0, \pi) - \text{int } H_0$ onto $W(\pi/2, \pi) - \text{int } H_{\pi/2}$ and fixes int $W(\pi, 2\pi)$. The set $W(0, \pi) \cap f(B^n)$ is separated
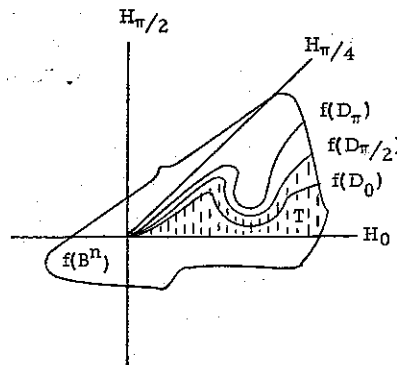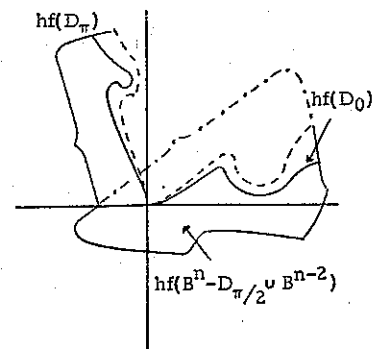

FIGURE 1   FIGURE 2
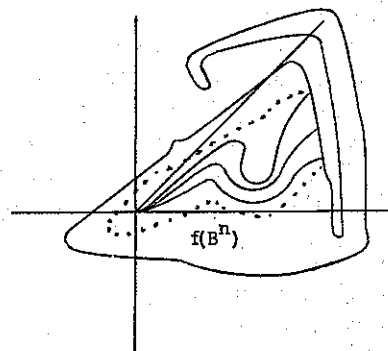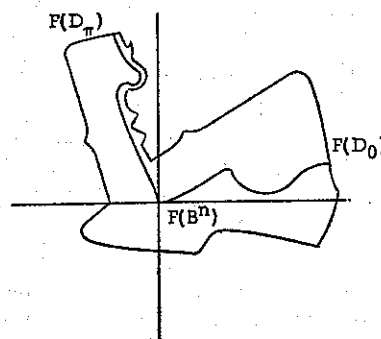
FIGURE 3   FIGURE 4

into two sets by $f(D_{\pi/2})$; let T denote the set containing $f(D_0)$. Then (see Figure 2) define an imbedding $h: f(B^n - D_{\pi/2} \cup B^{n-2}) \to R^n$ by

$$h(f(x)) = f(x) \quad \text{if } f(x) \in T,$$
$$= \bar{h}f(x) \quad \text{if } f(x) \notin T.$$

To ensure that h is an imbedding it may be necessary to trim away part of $f(B^n)$, still leaving a "ball-neighborhood" of $f(D_0)$ (in Figure 3, restricting to the dotted ball would eliminate the annoying feelers). Note that $hf = f$ on D_0 and $hf(D_\pi) \subset W(\pi/2, \pi)$.

We need to extend $hf | \tilde{W}(\pi, 2\pi)$ to an imbedding of B^n into R^n. We can assume that for some $\epsilon > 0$, $f(D_{2\pi-\epsilon}) \subset W(0, \pi/2)$, so then $hf = f$ on $D_{2\pi-\epsilon}$. Let g_1 be the homeomorphism of $B^n - D_{\pi/2} \cup B^{n-2}$ which fixes points outside $\tilde{W}(3\pi/4, 2\pi)$ and moves $D_{2\pi-\epsilon}$ to D_π. Let g_2: $hf(\tilde{W}(3\pi/4, 2\pi-\epsilon)) \to hf(\tilde{W}(3\pi/4, \pi))$ be the homeomorphism defined by $g_2 = hfg_1(hf)^{-1}$. Now define an imbedding $g: f(\tilde{W}(0, 2\pi-\epsilon) \to R^n$ by

$$g(x) = g_2(x) \quad \text{if } x \in hf(\tilde{W}(3\pi/4, 2\pi - \epsilon)),$$
$$= x \quad \text{otherwise.}$$

To make sure that g is well defined, it may be necessary to again shrink $f(B^n)$ towards B^{n-2} so that int $f(\tilde{W}(0, 2\pi-\epsilon)) \cap \partial hf(\tilde{W}(3\pi/4, 2\pi-\epsilon))$ $\subset hf(D_{3\pi/4})$. Let $i: \tilde{W}(0, \pi) \to \tilde{W}(0, 2\pi-\epsilon)$ and note that $gfi = hf$ on D_π. Then (see Figure 4), we can piece together gfi and hf to get an imbedding $F: B^n \to R^n$; specifically, let

$$F(x) = gfi(x) \quad \text{if } x \in \tilde{W}(0, \pi),$$
$$= hf(x) \quad \text{if } x \in \tilde{W}(\pi, 2\pi).$$

$F = f$ on D_0, so $F(D_0) \subset W(-\pi/2, \pi/2)$, and $F(D_\pi) = hf(D_\pi)$

$\subset W(\pi/2, 3\pi/2)$. Thus $F(B^{n-1})$ is "transverse" to $H_{\pi/2} \cup H_{3\pi/2}$, and that is the key to the proof. It allows us to find an isotopy making $F(D_\theta)$ tangent to H_θ at B^{n-2} for all θ. This isotopy is constructed in the latter part of the proof of Lemma 5.2 of [2]. Then a homeomorphism of R^n can be constructed which fixes D_π and takes $F(D_0)$ to D_0 (see the proof of Theorem 6.1 in [2]). Thus $(R^n, \beta_1 \cup \beta_2)$ is pairwise homeomorphic to $(R^n, D_\pi \cup D_0)$, finishing the proof.

REFERENCES

1. A. V. Černavskiĭ, *Singular points of topological imbeddings of manifolds and the union of locally flat cells*, Soviet Math. Dokl. 7 (1966), 433–436.

2. J. C. Cantrell and R. C. Lacher, *Some conditions for manifolds to be locally flat*, (to appear).

3. T. Hutchinson, *Two-point spheres are flat*, Notices Amer. Math. Soc. 14 (1967), 364, Abstract 644–18.

4. R. C. Kirby, *On the set of non-locally flat points of a submanifold of co-dimension one*, Ann. of Math.

5. R. C. Lacher, *Locally flat strings and half-strings*, Proc. Amer. Math. Soc. 18 (1967), 299–304.

UNIVERSITY OF CALIFORNIA, LOS ANGELES