ARTICLE EXTRAIT DES ACTES DU CONGRÈS INTERNATIONAL DES MATHÉMATICIENS

NICE - SEPTEMBRE 1970

ARTICLE FROM THE PROCEEDINGS OF THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

NICE - SEPTEMBER 1970

GAUTHIER-VILLARS ÉDITEUR
55, quai des Grands-Augustins, Paris 6e
1971

Actes, Congrès intern. Math., 1970. Tome 2, p. 79 à 84.

SOME CONJECTURES ABOUT FOUR-MANIFOLDS

by R.C. KIRBY

The theory of topological manifolds and their piecewise linear structures has reached a certain level of completeness in dimensions not equal to four, (see references [2]-[10]). We list below some theorems on existence and uniqueness of PL structures, TOP transversality, and TOP handlebodies which are unknown in dimension 4. There are some natural conjectures about these and other theories for dimension 4 ; we show that they are all related, and that in fact the Product Structure Theorem in dimension 4 would imply all of them.

The theory presented here is rather simple ; basically it assumes that the higher dimensional theory holds as much as possible in dimension 4, given Rohlins theorem [12] and the uniqueness of PL structures on 3-manifolds [11], both of which are anomalies. If this theory does not hold, it would seem that the correct theory must be much more complicated.

Q will always refer to a TOP manifold of dimension q, and C will be a closed subset. A property is said to hold *near* a closed set if it holds on a neighborhood of the closed set. "CAT" refers to either the PL or DIFF category. A CAT structure on a manifold is denoted by a capital Greek letter, e.g. Q_Σ, except that $I = [0, 1]$ and R^s denote the unit interval and Euclidean s-space *with* their usual linear structures.

Two PL structures Σ and Θ on Q which agree near C are equivalent if there exists a PL homeomorphism $h : Q_\Sigma \to Q_\Theta$ with h = identity near C. Σ and Θ are equivalent up to isotopy (resp. homotopy) if h is isotopic (resp. homotopic) modulo C to the identity homeomorphism.

Let $h : B^k \times R^n \to V^{k+n}$ be a homeomorphism of the unit k-ball cross n-space onto a PL manifold V^{k+n} such that h is PL near the boundary $S^{k-1} \times R^n$. "Straightening the k-handle h" means finding an isotopy $h_t : B^k \times R^n \to V$, $t \in [0, 1]$, such that $h_0 = h$, h_1 is PL near $B^k \times B^n$, and $h_t = h$ near $S^{k-1} \times R^n$ and outside a compact set.

A central theorem in the theory of TOP manifolds is the.

PRODUCT STRUCTURE THEOREM [5]. — *Let $q \geq 5$ or $q = 5$ if $\partial Q \subset C$. Let Σ_0 be a* CAT *structure near C. Let Θ be a* CAT *structure on $Q \times R^s$ which agrees with $\Sigma_0 \times R^s$ near $C \times R^s$. Then Q has a* CAT *structure Σ, extending Σ_0 near C, and $\Sigma \times R^s$ is concordant to Θ modulo $C \times R^s$.*

In fact, there is an ϵ-isotopy $h_t : Q_\Sigma \times R^s \to (Q \times R^s)_\Theta$ with h_0 = identity, h_1 a CAT *homeomorphism, and h_t = identity near $C \times R^s$, where $\epsilon : Q \times R^s \to (0, \infty)$ is a continuous function.*

This theorem is easy to prove for $q \leq 2$, and $q + s \geq 6$ or $q + s = 3$ or $q + s = 5$ if $\partial Q \subset C$, for then Q and $Q \times R^s$ have unique PL structures up to isotopy.

The Product Structure Theorem is known to fail for closed 3-manifolds ; the PL structures up to isotopy on $Q^3 \times R^n$, Q closed and $n \geqslant 2$, are classified by $H^3(Q\ ;\ Z_2) = Z_2$ but there is only one PL structure up to isotopy on Q^3. Moreover, these counterexamples are not valid just for equivalence *up to isotopy*, because $S^3 \times R^2$ has two PL structures which are not equivalent (PL homeomorphic) [2], [8].

Conjecture A_1. The Product Structure Theorem holds for $q = 5$ and $\partial Q \not\subset C$ or for $q = 4$ and $\partial Q \subset C$.

Conjecture A_2. The Product Structure Theorem holds for $q = 4$ and $\partial Q \not\subset C$ (respectively for $q = 3$) if ∂Q (respectively Q) has a handlebody decomposition with no 3-handles which are not in C.

Conjecture A_1 implies Conjecture A_2 (see Theorem 6). In fact Conjecture A_1 implies all the other conjectures in this paper. Note that Conjectures A_1 and A_2 imply that the Product Structure Theorem holds for all open manifolds with $C = \emptyset$.

To prove A_1, it would suffice to know that several 5-dimensional relative CAT s-cobordisms were CAT products [5]. Consider the CAT s-cobordism $(Z\ ;\ Y_0, Y_1)$ where $(Z\ ;\ Y_0, Y_1)$ is homeomorphic to either $(I\ ; 0\ , 1) \times B^k \times T^{4-k}$ or

$$(I\ ; 0\ , 1) \times B^k \times S^{3-k} \times R$$

and the homeomorphism is CAT near $1 \times Y_1$ and near $I \times \partial Y_1$. Then Conjecture A_1 is equivalent to knowing $(Z\ ;\ Y_0, Y_1)$ is CAT homeomorphic to $(I\ ; 0\ , 1) \times Y_1$ relative to $1 \times Y_1$ and $I \times \partial Y_1$.

CONCORDANCE-IMPLIES-ISOTOPY THEOREM [2], [5], [8]. — *Let $q \geqslant 6$ or $q = 5$ if $\partial Q \subset C$. Let Γ be a* CAT *structure on $I \times Q$, and $0 \times \Sigma$ its restriction to $0 \times Q$. Suppose $\Gamma = I \times \Sigma$ near $I \times C$. Then there exists an ϵ-isotopy $h_t : I \times Q_\Sigma \to (I \times Q)_\Gamma$, $t \in [0\ , 1]$, such that $h_0 = $ identity, $h_t = $ identity on $0 \times Q$ and near $I \times C$, and h_1 is a* CAT *homeomorphism, where $\epsilon : I \times Q \to (0\ , \infty)$ is continuous.*

This theorem is well known [11] for $q \leqslant 2$. It also fails in dimension 3 or 4 because if it was true in all dimensions, it would imply the Product Structure Theorem in all dimensions [5].

Conjecture B_1. Concordance implies isotopy for $q = 5$ and $\partial Q \not\subset C$ or $q = 4$ and $\partial Q \subset C$.

Conjecture B_2. Concordance implies isotopy for $q = 4$ and $\partial Q \not\subset C$ (respectively for $q = 3$) if ∂Q (respectively Q) has a handlebody decomposition with no 2-handles that are not in C.

THEOREM 1. — *Conjecture A_1 is equivalent to Conjecture B_1.*

Proof. — In [5], it is shown that the Concordance-implies-isotopy Theorem plus the Annulus Theorem are together equivalent to the Product Structure Theorem. The same method of proof gives Theorem 1 since Conjecture B_1 implies the Annulus Conjecture in dimension 4.

CLASSIFICATION THEOREM [3] [6] [8] [10]. — *Let $q \geqslant 6$, or $q = 5$ and $\partial Q \subset C$. The homotopy classes of reductions of the stable tangent bundle of Q to a* CAT *bundle, modulo C, correspond bijectively to isotopy classes of* CAT *structures on Q agreeing with a given* CAT *structure Σ_0 near C.*

This theorem fails for closed 3-manifolds, for there exist two reductions but only one CAT structure on Q^3.

However the non-stable Classification Theorem [6], [8], [9], (for reductions of the tangent bundle itself) holds for $q \leqslant 3$ as well as $q \geqslant 6$, or $q = 5$ and $\partial Q \subset C$.

THEOREM 2. — *Conjecture A_1 implies that the stable Classification Theorem holds also for $q = 5$ or $q = 4$ and $\partial Q \not\subset C$, and that the non-stable Classification Theorem holds without dimensional restriction.*

Proof. — The only ingredients in the proof of the stable Classification Theorem requiring dimensional restrictions are the Product Structure Theorem and the Concordance-implies-isotopy Theorem ; the dimensional restrictions are lowered by one in Conjectures A_1 and B_1. The non-stable version uses immersion theory and requires only Conjecture B_1 to hold in all dimensions.

THEOREM 3. — *Conjecture A_1 implies*

(i) *$h : B^k \times R^{4-k} \to V^4$ can be "straightened" if $k \neq 3$,*

(ii) *there exists $h' : B^3 \times R \to V$ which cannot be straightened, and*

$$h' \times \mathrm{id} : B^3 \times R^2 \to V \times R$$

corresponds to the non-zero element in $\pi_3(\mathrm{TOP}_5, \mathrm{PL}_5) = Z_2$.

(iii) $\pi_4(\mathrm{TOP}_4, \mathrm{PL}_4) = \begin{cases} 0 & k \neq 3 \\ Z_2 & k = 3 \end{cases}$

(iv) $\pi_3(\mathrm{TOP}_3, \mathrm{PL}_3) = 0 \to \pi_3(\mathrm{TOP}_4, \mathrm{PL}_4) \xrightarrow[s]{\cong} \pi_3(\mathrm{TOP}_5, \mathrm{PL}_5) \xrightarrow{\cong} \cdots$ *where s is the stabilization map.*

Note that similar statements are true in dimensions > 4.

Proof. — (i) h pulls back a PL structure from V onto $B^k \times R^{4-k}$, say Σ, which agrees with the standard structure near the boundary. $\Sigma \times R$ is equivalent (modulo boundary) up to isotopy with the standard structure because $h \times \mathrm{id}$ can be straightened [5]. By Conjecture A_1, Σ is equivalent (modulo boundary) up to isotopy with the standard structure, so we compose this isotopy with h to straighten h.

(ii) $B^3 \times R^2$ has an exotic PL structure which agrees with the standard one near the boundary, so by Conjecture A_1, $B^3 \times R$ has an exotic PL structure Σ' which is standard near $S^2 \times R$. Let $h' = \mathrm{id} : B^3 \times R \to (B^3 \times R)_{\Sigma'}$.

(iii) and (iv) follow as in [2], or [3] or [7].

It is known that if Σ_0 is a PL structure near C, then Q has a PL structure Σ extenting Σ_0 near C if $q \leqslant 3$ or if $q \geqslant 6$ and $H^4(Q, C; Z_2) = 0$ or if $q = 5$, $\partial Q \subset C$ and $H^4(Q, C; Z_2) = 0$. The remaining cases are taken care of by

THEOREM 4. — *Conjecture A_1 implies*

(i) *If $q = 4$ and $\partial Q \subset C$ or $q = 5$ and $\partial Q \not\subset C$, then Q has a* PL *structure Σ extending Σ_0 near C if $H^4(Q, C; Z_2) = 0$,*

(ii) *If $q = 4$ and $\partial Q \not\subset C$ then Σ exists if $H^4(Q, C \cup \partial Q; Z_2) = 0$.*

Proof. — (i) The theorem follows immediately because the stable Classification Theorem holds in these dimensions (see Theorem 2).

(ii) ∂Q has a unique PL structure [11] so we extend Σ_0 to a neighborhood of $\partial Q \cup C$ and use (i).

Also it is known that if Q has a PL structure Σ, then the PL structures, agreeing with Σ near C, are unique up to isotopy if $q \leqslant 3$, and are classified up to isotopy by $H^3(Q\,,C\,;Z_2)$ if $q \geqslant 6$, or if $q = 5$ and $\partial Q \subset C$.

THEOREM 5. — *Conjecture* A_1 *implies*

(i) *the isotopy classes of* PL *structures* mod C *are still classified by* $H^3(Q\,,C\,;Z_2)$ *if* $q = 5$ *and* $\partial Q \not\subset C$ *or if* $q = 4$ *and* $\partial Q \subset C$.

(ii) *the isotopy classes of* PL *structures* mod C *are classified by*

$$H^3(Q\,,C \cup \partial Q\,;Z_2)$$

if $q = 4$ *and* $\partial Q \not\subset C$.

Proof. — See the proof of Theorem 4.

THEOREM 6. — *Conjecture* A_1 *implies Conjecture* A_2 *and Conjecture* B_1 *implies Conjecture* B_2.

Thus we have

$$\begin{array}{ccc} \text{Conjecture } A_1 & \Leftrightarrow & \text{Conjecture } B_1 \\ \Downarrow & \searrow & \Downarrow \\ \text{Conjecture } A_2 & \Rightarrow & \text{Conjecture } B_2 \end{array}$$

Proof. — These Conjectures depend on the following conjecture which is a handle version of concordance-implies-isotopy (see §§ 3, 4, 5 of [5]).

Let $H : (I\,,0) \times B^k \times R^n \to (X\,,V)$ be a homeomorphism, which is CAT near $(1 \times B^k \times R^n) \cup (I \times S^{k-1} \times R^n)$, onto a CAT manifold X where V is a codimension one, CAT locally flat, submanifold. Then there exists a pairwise isotopy $H_t : (I\,,0) \times B^k \times R^n \to (X\,,V)\,, t \in [0\,,1]$, with $H_0 = H\,, H_1 =$ CAT homeomorphism on $I \times B^k \times B^n$, and $H_t = H$ near $(1 \times B^k \times R^n) \cup (I \times S^{k-1} \times R^n)$ and outside a compact set.

The Concordance-implies-isotopy Theorem gives an H_t if $k + n \neq 3$, 4. Conjecture A_1 implies the cases $k + n = 4$, and $k + n = 3$ with $k \neq 2$. But if $k = 2\,, n = 1$, then some H cannot be straightened. The implications in Theorem 5 can be derived from this.

THEOREM 7 (TOP transversality). — *Let* $\xi^n = (E(\xi^n) \xrightarrow{\pi} X)$ *be an n-plane bundle over a topological space X and let* $f : M^m \to E(\xi^n)$ *be a continuous function. Then if* $m \neq 4$ *and* $m - n \neq 4$, *f is homotopic to a map* f_1 *which is transverse to the 0-section of* ξ *(this means that* f^{-1} *(0-section) is an* $(m-n)$*-manifold P with normal bundle in M equal to* $(\pi f_1 | P)^*(\xi)$*). Moreover, if f is already transverse near a closed set C in M, then the homotopy equals f near C.*

THEOREM 8. — *Conjecture* A_1 *implies that* TOP *transversality holds in all dimensions.*

Proof. — The proof of Theorem 7 (see [2]) uses only the Product Structure Theorem for open manifolds with $C = \emptyset$ which follows in all cases from Conjecture A_1.

THEOREM 9 (TOP handlebody structures). — *If* $m \geqslant 6$, *then* M^m *is a* TOP *handlebody* (*if* $m = 6$ *and* $\partial M \neq \emptyset$, *then we obtain* M *by adding handles to* ∂M). *Equivalently,* M *admits a Morse function* $f : M \to R$ (*that is,* f *is locally of the form* $x_1^2 + \cdots + x_\lambda^2 - x_{\lambda+1}^2 - \cdots - x_m^2$.

THEOREM 10. — *Conjecture* A_1 *implies that all* 5*-manifolds are* TOP *handlebodies. It follows that there is a* 4*-manifold which is not a* TOP *handlebody.*

Proof. — See [2] [13]. Note that if $\dim(\partial M) = 4$, then we give M a handlebody structure by adding handles to ∂M. The boundary of a 5-dimensional TOP handlebody is not necessarily a TOP-handlebody.

It is known [7] that $Z = \pi_4(B_{\mathrm{PL}}) \to \pi_4(B_{\mathrm{TOP}}) = Z$ is multiplication by two. If ξ is the generator of $\pi_4(B_{\mathrm{TOP}})$, then ξ represents a TOP n-plane bundle

$$\xi^n = (E(\xi^n) \xrightarrow{\pi} X)$$

which is fiber homotopy trivial. If $f : E(\xi^n) \to R^n$ is the trivialization, then using TOP transversality (Theorem 8), $M^4 = f^{-1}(0)$ is a closed, almost parallelizable 4-manifold. Furthermore, the identification of $\pi_4(B_{\mathrm{TOP}})$ with Z tells us that index $(M^4) = 8$, because the generator of $\pi_4(B_{\mathrm{PL}})$ corresponds to a PL 4-manifold of index 16.

M^4 cannot be PL by Rohlins Theorem [12]. Therefore M^4 is not a TOP handlebody. (Any 4-dimensional TOP handlebody is PL since the attaching maps are 3-dimensional imbeddings and can be straightened [11]). A more explicit construction of such an M^4 appears in [13].

REFERENCES

[1] KIRBY R. — Stable homeomorphisms and the annulus conjacture, *Ann. of Math.*, 89, 1969, p. 575-582.

[2] KIRBY R. — Lectures on triangulations of manifolds, *Mimeo. notes*, U.C.L.A., Los Angeles, 1969.

[3] KIRBY R. and SIEBENMANN L.C. — On the triangulation of manifolds and the Hauptvermuntung, *Bull. Amer. Math. Soc.*, 75, 1969, p. 742-749.

[4] KIRBY R. and SIEBENMANN L.C. — *Notices Amer. Math. Soc.*, 16, 1969, p. 587, 695, 848.

[5] KIRBY R. and SIEBENMANN L.C. — *Deformation of smooth and piecewise linear manifold structures*, to appear.

[6] KIRBY R. and SIEBENMANN L.C. — *Classification of smooth and piecewise linear manifold structures using the product structure theorem*, to appear.

[7] KIRBY R. and SIEBEMMANN L.C. — To appear (sequel to [5], [6]).

[8] KIRBY R. and SIEBENMANN L.C. — Some theorems on topological manifolds, to appear in Proceedings of summerschool on manifolds, Amsterdam, 1970, *Lecture notes on Math.*, Springer-Verlag.

[9] LASHOF R.K. — The immersion approach to triangulation, Topology of Manifolds, (edited by Cantrell and Edwards), Markham, Chicago, 1970, p. 52-57.

[10] Lashof R.K. and Rothenberg M. — Triangulation of manifolds I, II, *Bull. Amer. Math Soc.*, 75, 1969, p. 750-757.

[11] Moise E. — Affine structures on 3-manifolds, *Ann. of Math.*, 56, 1957, p. 96-114.

[12] Rohlin V.A. — A new result in the theory of 4-dimensional manifolds, *Doklady*, 8, 1957, p. 221-224.

[13] Siebenmann L.C. — Disruption of low dimensional handlebody theory by Rohlins Theorem, (edited by Cantrell and Edwards), Markham, Chicago, 1970, p. 57-77.

U.C.L.A
Dept. of Mathematics,
Los Angeles, California 90024
U.S.A