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On the set of non-locally flat points of a submanifold of codimension one*

By Robion C. Kirby

THEOREM 1. *Let $f: M^{n-1} \to N^n$ be an imbedding of a topological $(n-1)$-manifold M without boundary into a topological n-manifold N without boundary. Let E be the set of points of M at which f is not locally flat. Then, if $n \geqq 4$, E cannot be a non-empty subset C^* of a Cantor set where C^* is tame in M and $f(C^*)$ is tame in N.*

Chernavskii [8] and Hutchinson [10] have announced results similar to this, proved using engulfing techniques.

If $\partial M \neq \emptyset$ and $E \subset \partial M$, then the same conclusion holds. Let $\lambda: M \to [0, 1]$ be a continuous function which is zero on ∂M and non-zero elsewhere. Consider the manifold $\tilde{M}$ in $M \times [-1, 1]$ defined by $\tilde{M} = \{(x, t) \mid -\lambda(x) \leqq t \leqq \lambda(x)\}$. Since f is flat on the interior of M, f extends to an imbedding $f': \tilde{M} \to N$ for which $f' \mid \partial\tilde{M}$ is locally flat except at E. Since E is tame in $\partial\tilde{M}$ if it is tame in M, and since $\partial\partial\tilde{M} = \emptyset$, we may apply Theorem 1 to see that E cannot be a subset of a Cantor set.

Suppose $p \in E$ is isolated in E. Choose M' to be a neighborhood of p with $p = E' = E \cap M'$, and let α be a flat arc in M' through p. $f(\alpha - p)$ is flat, so by [7], $f(\alpha)$ is flat, and therefore p and $f(p)$ are tame. Applying Theorem 1 to $f \mid M' : M' \to N$, we see that $p \notin E' \subset E$. Thus E has no isolated points and hence must be uncountable, since a closed countable subset of a manifold must contain isolated points. This fact also follows from the theorem that the union of flat cells is flat in codimension one (see [13] or [14]).

The fact that $f(C^*)$ is tame in N cannot be removed from the hypothesis of Theorem 1. A counter-example in R^n could be constructed from Blankinship's arc [1]. This arc is wild at a Cantor set, but lies in an $(n-1)$-sphere which is locally flat except at the Cantor set; furthermore, this Cantor set is tame in the $(n-1)$-sphere.

COROLLARY. *Theorem 1 holds in the special case that $M^{n-1} = S^{n-1}$ and $N^n = R^n$ or S^n.*

THEOREM 2. *The corollary is equivalent to Theorem 1.*

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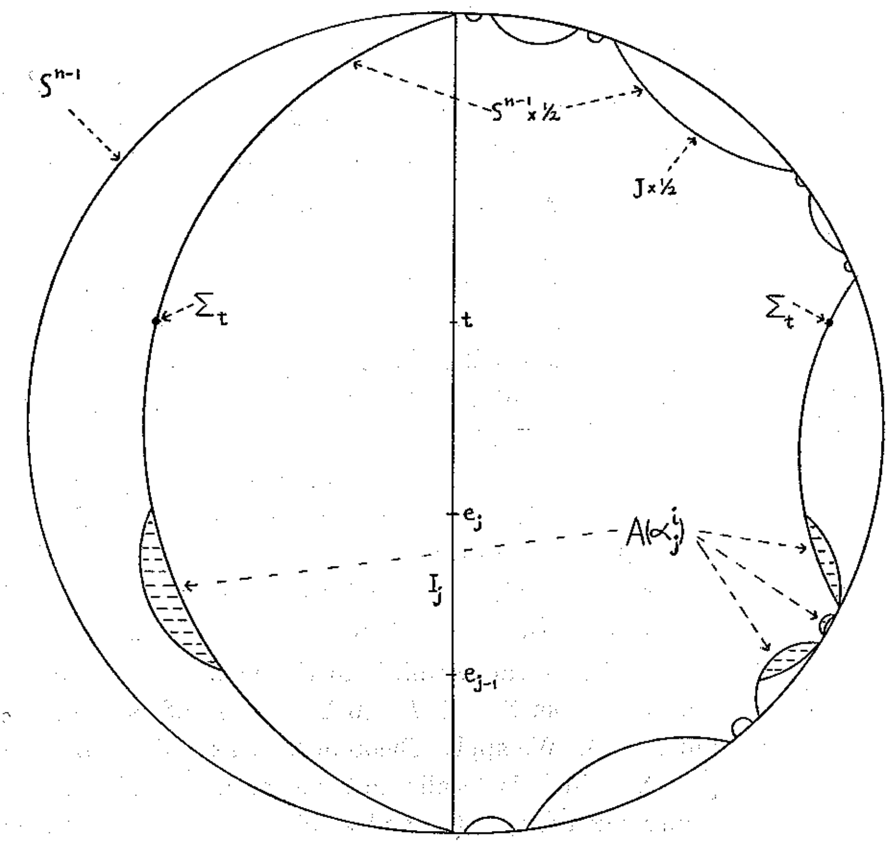

Figure 1

PROOF OF THE COROLLARY. $f(S^{n-1})$ separates R^n into a bounded open set Z "inside" $f(S^{n-1})$ and an unbounded set "outside". Since C^* is tame in S^{n-1}, there is a homeomorphism of S^{n-1} taking C^* onto C, so we assume $C^* = C$. $f \mid (S^{n-1} - C)$ is locally flat, so it is flat, so we can assume f imbeds the pinched annulus A into R^n with $f(S^{n-1} \times 1)$ in $\bar{Z}$.

Since $f \mid (J - C)$ is locally flat and $f(C)$ is tame we can conclude from [4] that $f(J)$ is flat in R^n.

Let $L = f(J \times 1/2)$; it is flat. Denote points of R^n lying on L by t_L, $t \in [-1, 1]$.

The idea of the proof is to find a map $K: R^n \to R^n$ such that K is a homeomorphism outside $f(S^{n-1} \times 1/2)$, is the identity outside $f(S^{n-1})$, and collapses $f(S^{n-1} \times 1/2)$ to L, taking $f(\Sigma_t)$ to t_L (see (D) below). This "fills up" $\bar{Z}$, which is then shown to be an n-ball (see (E) below). Then f will be locally flat at C, completing the proof of the corollary.

(A) Z is homeomorphic to R^n.

PROOF. Since $f(J)$ is flat, we may shrink it to a point p by a map $g: R^n \to R^n$ where g is a homeomorphism off $f(J)$. Then $gf(S^{n-1})$ is an $(n-1)$-sphere which is locally flat except at p. By [5], $gf(S^{n-1})$ bounds an n-ball. In particular, the interior $g(Z)$, is homeomorphic to R^n, so Z is also homeomorphic to R^n.

(B) Given $\varepsilon > 0$, there exists a homeomorphism $G: R^n \to R^n$ for which

(1) $G =$ identity outside $f(S^{n-1})$,

(2) $Gf(S^{n-1} \times 1/2) \subset N_\varepsilon(L)$.

Furthermore, this holds for any other C' and A'.

PROOF. Let s be the map which shrinks L to a point p and is a homeomorphism elsewhere. $sf(S^{n-1} \times 1/2)$ is an $(n-1)$-sphere with only one non-locally flat point p. Then it bounds an n-ball by [6]. So each point $x \in sf(S^{n-1} \times 1/2)$ is joined by a ray r_x to p. Under s^{-1} (not defined at p) these rays are taken to rays called $s^{-1}r_x$, which may not converge to any point of L, but do get arbitrarily close to L. Using these rays, we may slide $f(S^{n-1} \times 1/2)$ into $N_\varepsilon(L)$, fixing L. The result of this slide is called G.

(C) For any $t \in (-1, 1) - C$ and hence for any $t \in D$, there exists a map $H: R^n \to R^n$ which satisfies

(1) $Hf(\Sigma_t) = t_L$,

(2) $H =$ identity outside $f(S^{n-1})$,

(3) H is a homeomorphism outside $f(S^{n-1} \times 1/2)$ and on a neighborhood of $f(S^{n-1} \times (1/2) - \Sigma_t)$.

It is implied by (3) that Hf is a locally flat imbedding of $(S^{n-1} \times 1/2) - C - \Sigma_t$. Also, (C) holds for other C' and A'.

PROOF. The set $S_t^{n-2} \times (0, 1/2]$ in A may be identified with $B^{n-1} - 0$. Then $f(B^{n-1} - 0) = f(S_t^{n-2} \times (0, 1/2])$ lies in Z as a closed, flat submanifold. Adding ∞ to $Z = R^n$, we obtain Σ^n. Letting $f(0) = \infty$, f imbeds B^{n-1} and is flat on $B^{n-1} - 0$. By Theorem 3, f extends to an imbedding $f^*: \Sigma^{n-1} \to \Sigma^n$, flat on $\Sigma^{n-1} - \{0, \infty\}$. We have $f^*(\Sigma^{n-1} - 0) \subset Z$. We can ensure that

$$f^*(\Sigma^{n-1} - 0) \cap f(A_{1/2}) = f^*(B^{n-1} - 0) = f\left(S_t^{n-2} \times \left(0, \frac{1}{2}\right]\right)$$

by pushing $f^*(\Sigma^{n-1} - B^{n-1})$ off $f(A_{1/2})$ using the collaring of $f(S^{n-1})$.

Now let α be the shortest geodesic in Σ^{n-1} joining $(f^*)^{-1}(t_L)$ to ∞. Then $f^*(\alpha)$ is locally flat mod $f(\infty)$, hence flat. Let $s: Z \to Z$ be the map which shrinks $f^*(\alpha)$ to t_L, is the identity on $f(A_{1/2})$, and is a homeomorphism off $f^*(\alpha)$. Then $f(S_t^{n-2} \times 1/2) = f(\Sigma_t)$ bounds $sf^*(\Sigma^{n-1} - \operatorname{int} B^{n-1})$, an $(n-1)$-ball which is flat except at t_L, and whose interior misses $f(A_{1/2})$. It is now easy to construct H by shrinking this $(n-1)$-ball to t_L.

For there is an arc $A \subset f(B^{n-1})$ (the image of a radius) from $f(0)$ to a point $b \in f(\partial B^{n-1})$, which is locally flat mod $f(0)$, hence flat. Then there exists a map $s\colon \Sigma^n \to \Sigma^n$ which is the identity on $f(\partial B^{n-1})$, a homeomorphism off A and shrinks A to b. Then $sf(B^{n-1})$ is an $(n-1)$-ball with boundary $f(\partial B^{n-1})$ which is locally flat except at the point b in its boundary. By [11] this ball is flat. Therefore $f(\partial B^{n-1})$ is unknotted in Σ^n, so a space homeomorphism takes it onto S^{n-2}. Furthermore we can assume $f(0) = 0$.

Let $j\colon \Sigma^{n-1} \to 2\widetilde{B}^n - \partial B^{n-1}$ be a map satisfying

(i) $j =$ identity in a neighborhood of 0,

(ii) $j(\infty) = 0$,

(iii) $j \mid (\Sigma^{n-1} - \{0, \infty\})$ is an imbedding, and

(iv) if ρ is a compactified ray in Σ^{n-1} beginning at 0, then $j(\rho)$ lies in the plane P determined by X_n and ρ and winds once around ∂B^{n-1} as in Figure 2. In particular we require that $j(\rho)$ "represents" a generator of

$$H_1(2\widetilde{B}^n - \partial B^{n-1}) = Z\ .$$

To ensure local flatness later on, we assume that j is extended to a product neighborhood N of Σ^{n-1}, pinched at 0 and ∞, such that $j \mid N - \{0, \infty\}$ is an imbedding into $2\widetilde{B}^n - \partial B^{n-1}$. Now we wish to "unwind" $fj(N)$, rotating

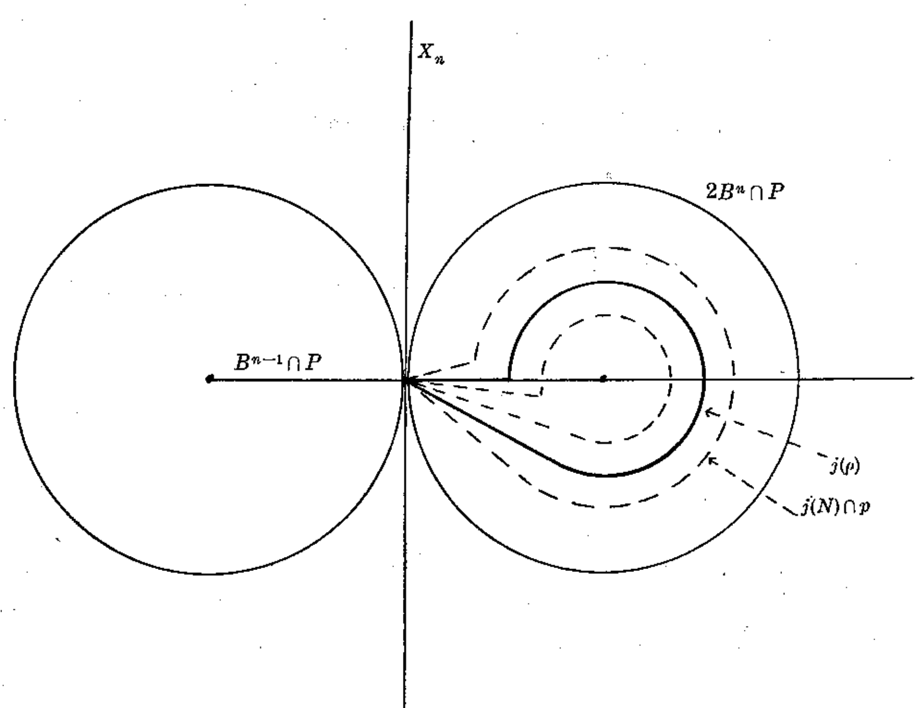

Figure 2

around S^{n-2}, so that $fj(0)$ and $fj(\infty)$ no longer coincide; then Σ^{n-1} will be flatly imbedded except at 0 and ∞ and will contain a neighborhood of $f(0)$ in $f(B^{n-1})$.

There is a natural homeomorphism of $\operatorname{int} B^{n-1} \times S^1$ onto $\Sigma^n - S^{n-2}$ taking $0 \times S^1$ onto the compactified X_n-axis and $\operatorname{int} B^{n-1} \times 0$ onto $\operatorname{int} B^{n-1}$ (here S^1 is $[0, 2\pi]$ with $0 = 2\pi$). Identifying $\Sigma^n - S^{n-2}$ with $\operatorname{int} B^{n-1} \times S^1$ in this way, let $p_1: \Sigma^n - S^{n-2} \to \operatorname{int} B^{n-1}$ and $p_2: \Sigma^n - S^{n-2} \to S^1$ be the projections. Let $q: R^1 \to S^1$ be the universal covering space (where $q(x) \equiv x \bmod 2\pi$). Since N is simply connected, the map $p_2 fj: N \to S^1$ lifts to a unique map $\lambda: N \to R^1$ satisfying $q\lambda = p_2 fj$ and $\lambda(0) = 0$. In effect, λ assigns to each point in N a "winding number" around S^{n-2}.

We now show that $\lambda(\infty) \neq 0$. Let ρ be a compactified ray in Σ^{n-1} as above, so ρ is an arc with end points 0 and ∞. Recalling that $f(\partial B^{n-1}) = S^{n-2}$, $fj(\rho)$ represents a generator of $H_1(f(2\widetilde{B}^n) - S^{n-2})$, since $j(\rho)$ represents a generator of $H_1(2\widetilde{B}^n - S^{n-2})$. By excision on the pair $(\Sigma^n, f(2\widetilde{B}^n))$, we see that $H_i(\Sigma^n - S^{n-2}, f(2\widetilde{B}^n) - S^{n-2}) = 0$, $(i = 1, 2)$, and hence the inclusion $f(2\widetilde{B}^n) - S^{n-2} \subset \Sigma^n - S^{n-2}$ induces an isomorphism of first homology groups. Thus $fj(\rho)$ also represents a generator of $H_1(\Sigma^n - S^{n-2})$. Finally, $p_2: \Sigma^n - S^{n-2} \to S^1$ is a homotopy equivalence, so $p_2 fj(\rho)$ represents a generator of $H_1(S^1)$. Thus $\lambda(\infty) = \pm 2\pi$.

To unwind the map fj, let $\alpha: R^1 \to (-\pi, \pi)$ be a homeomorphism which is the identity on $(-\pi/2, \pi/2)$, and define $G: N \to \Sigma^n - S^{n-2} = \operatorname{int} B^{n-1} \times S^1$ by $G(z) = (p_1 fj(z), q\alpha\lambda(z))$. The verification that G is an imbedding is routine, noting that $\lambda(0) \neq \lambda(\infty)$. If $z \in B^{n-1}$ is sufficiently close to 0, then $p_1 fj(z) = p_1 f(z)$ and $q\alpha\lambda(z) = q\lambda(z) = p_2 fj(z) = p_2 f(z)$. Thus $g = G \mid \Sigma^{n-1}$ agrees with f near 0 and is locally flat off $0, \infty \in \Sigma^{n-1}$, completing the proof.

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