

## Akbulut's corks and h-cobordisms of smooth, simply connected 4-manifolds

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**Theorem:** Let  $M^5$  be a smooth 5-dimensional h-cobordism between two simply connected, closed 4-manifolds,  $M_0$  and  $M_1$ . Then there exists a sub-h-cobordism  $A^5 \subset M^5$  between  $A_0 \subset M_0$  and  $A_1 \subset M_1$  with the properties:

- (1):  $A_0$  and hence  $A$  and  $A_1$  are compact contractible manifolds, and
- (2):  $M - \text{int}A$  is a product h-cobordism, i.e. it is diffeomorphic to  $(M_0 - \text{int}A_0) \times [0, 1]$ .

This theorem first appeared in a preprint of Curtis & Hsiang in fall 1994. Soon after, much shorter proofs were found by Freedman & Stong [3], Matveyev [9], and Z. Bizaca. The following improvements were also shown:

**Addenda:** The h-cobordism  $A$  can be chosen so that,

- (A):  $M - A$  (and hence each  $M_i - A_i$ ) is simply connected (Freedman & Stong) [3],
- (B):  $A$  is diffeomorphic to  $B^5$  (Bizaca, Kirby) (but not, of course, preserving the structure of the h-cobordism),
- (C):  $A_0 \times I$  and  $A_1 \times I$  are diffeomorphic to  $B^5$  [9],
- (D):  $A_0$  is diffeomorphic to  $A_1$  by a diffeomorphism which, restricted to  $\partial A_0 = \partial A_1$ , is an involution [9].

**Corollary:** Any homotopy 4-sphere,  $\Sigma^4$ , can be constructed by cutting out a contractible 4-manifold,  $A_0$  from  $S^4$  and gluing it back in by an involution of  $\partial A_0$ .

**Remark:** Since there are many examples of non-trivial h-cobordisms (the first ones were discovered by Donaldson [4]), there are as many examples of non-trivial, rel boundary, h-cobordisms  $A$ . However these  $A$  are delicate objects; their non-triviality vanishes when a trivial h-cobordism is added. That is, if we add  $A_0 \times I$  to  $A$  along  $\partial A_0 \times I$ , then it follows from the Addenda that we have an h-cobordism between  $S^4$  on the bottom as well as  $S^4$  on the top; thus the h-cobordism is the trivial  $S^4 \times I$ .

1. The first example of a non-trivial h-cobordism,  $A$ , on  $B^5$  was found by Akbulut [1]. It is the prototype of the h-cobordisms  $A$  in the theorem, and it seems appropriate to call such h-cobordisms *Akbulut's corks*, for any exotic h-cobordism can be constructed from the product h-cobordism by pulling out a *cork* and putting it back in with a *twist*, (which preserves  $A_0$  but not the structure of the h-cobordism).

Akbulut constructs a homology  $S^2 \times S^2$ -point, called  $A_{1/2}$ , by adding two 2-handles to the symmetric link  $L$  of unknots drawn in Figure 1. Because this link  $L$  is symmetric, there

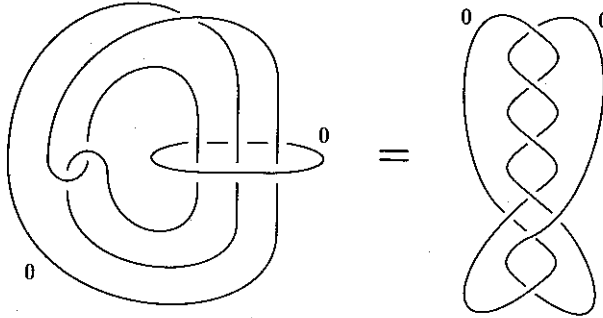


Figure 1.

is an involution  $\tau : \partial A_{1/2} \rightarrow \partial A_{1/2}$ , which extends to a diffeomorphism  $\bar{\tau} : A_{1/2} \rightarrow A_{1/2}$  which switches the 2-handles (to extend  $\tau$  over the 0-handle of  $A_{1/2}$ , one can either cone the involution on  $S^3$  obtaining an involution which is not smooth at the cone point, or extend over the 0-handle using the fact that the involution on  $S^3$  is diffeotopic to the identity).

Since the components of  $L$  are 0-framed unknots, one can trade either of the 2-handles (but not both for  $L$  is not the unlink) for a 1-handle, obtaining  $A_0$  or  $A_1$  respectively. (Recall that adding a 2-handle to a 0-framed unknot gives  $S^2 \times B^2$ , adding an orientable 1-handle to  $B^4$  gives  $B^3 \times S^1$ , and both have boundary  $S^2 \times S^1$ ; thus one may change the 4-manifold by trading 2-handles for 1-handles, or vice versa, and then adding all other handles to the  $S^2 \times S^1$  boundary as before the trade.) The new 1-handles are denoted by the same unknot but with a dot on it, which means that any arcs going through the dotted circle actually go over a 1-handle. This can be seen by observing that a dotted circle means: remove the obvious, properly imbedded, 2-ball in the 0-handle leaving  $S^1 \times B^3$ , which is a 0-handle union a 1-handle. This operation does not change boundaries, so  $\partial A_{1/2} = \partial A_0 = \partial A_1$ .

Akbulut proves (using a long series of handle moves culminating in an application of Donaldson's invariants) that the identity map,  $id : \partial A_0 \rightarrow \partial A_1$ , does not extend to a diffeomorphism of  $A_0$  to  $A_1$ .

The notation  $A_0, A_{1/2}, A_1$ , suggests there is an h-cobordism lurking about, and this is correct. The operation of trading 2-handles for 1-handles can also be done by adding 3-handles in the right way. Each component of  $L, K_0$  or  $K_1$ , determines a 2-sphere,  $S_0$  or  $S_1$ , composed of the core of the 2-handle and the obvious slice disk that  $K_i$  bounds in the 0-handle. Each 2-sphere has a trivial normal bundle (because of the 0-framing), and  $S_0 \cap S_1$  is three points (algebraically one). To construct an h-cobordism  $A$ , start

with  $A_{1/2} \times [1/2 - \epsilon, 1/2 + \epsilon]$  and add a 3-handle to  $S_0 \times (1/2 - \epsilon)$  and a 3-handle to  $S_1 \times (1/2 + \epsilon)$ . The new boundary on the bottom will be  $A_0$  because  $S_0 \times B^2$  has been removed from  $A_{1/2} \times (1/2 - \epsilon)$ , thus removing the 2-handle and the slice disk which has the effect of switching the 2-handle to a 1-handle. Similarly for  $A_1$ . The structure of the h-cobordism  $A$  is to add a 2-handle to  $A_0$  (the 3-handle turned upside down) and the 3-handle to  $S_1$ .  $A$  must be non-trivial because it is a product over  $\partial A_0$ , describing  $id : \partial A_0 \rightarrow \partial A_1$ , and Akbulut showed this cannot extend to a diffeomorphism from  $A_0$  to  $A_1$ .

There is a natural generalization of Akbulut's cork. Let the 0-handle in  $A_{1/2}$  be replaced by a contractible 4-manifold  $B_{1/2}$ . Suppose  $D$  is a collection of  $2n$  properly imbedded 2-balls,  $D_{0,i} \cup D_{1,i}, i = 1 \dots n$ , in  $B_{1/2}$ , and suppose that  $D_{0,i} \cap D_{0,j} = \emptyset = D_{1,i} \cap D_{1,j}$  for all  $i, j \in 1, \dots, n$ , and that algebraically  $D_{0,i} \cap D_{1,j} = \delta_{ij}$ . Then we can form  $A_{1/2}$  by adding 2-handles with 0-framings to each  $\partial D_{0,i}$  and  $\partial D_{1,i}, i = 1, \dots, n$ . This produces obvious 2-spheres with trivial normal bundles,  $S_{0,i}$  and  $S_{1,i}, i = 1, \dots, n$ . Then we form our h-cobordism  $A$  by adding  $n$  3-handles below and  $n$  3-handles above to the  $S_{0,i}$ 's and the  $S_{1,i}$ 's respectively.

We can conjecture that the product structure on the sides of  $A$ , namely  $\partial A_0 \times I$ , extends over  $A$  iff  $D$  is concordant in  $B_{1/2} \times I$  to the  $\partial$ -connected sum of  $n$  copies of  $B^2 \times 0 \cup 0 \times B^2$  in  $B^4 \subset B_{1/2}$ .

2. Here is a proof of the Theorem. The exposition is not particularly original, but gains by organizing all the steps into a whole rather than having them split into the two papers [3] and [9].

We begin with a Morse function  $f : (M, M_0, M_1) \rightarrow (I, 0, 1)$  and its associated handlebody structure which adds  $k$ -handles,  $0 \leq k \leq 5$ , to  $M_0$ . We can cancel all 0-handles and 5-handles since  $M$  is connected. We can cancel all 1-handles and 4-handles (at the cost of new 2 and 3-handles) just as Smale did in the original proof of the higher dimensional h-cobordism theorem ([10] Lemma 6.15). Alternatively, we may have the h-cobordism provided by Wall (see [11], [5] Chapter 9) which begins with homotopy equivalent, simply connected closed, smooth 4-manifolds  $M_0$  and  $M_1$  and constructs  $M$  using only 2 and 3-handles. (This involves no loss of generality because any two h-cobordisms between  $M_0$  and  $M_1$  are diffeomorphic [7, 8].)

We can assume that  $M_{1/2} = f^{-1}(1/2)$  has all the 2-handles below and all the 3-handles above. Note that  $M_{1/2}$  is 1-connected which is always true of the upper boundary when 2-handles are attached to a {simply connected 4-manifold}  $\times I$ . Each 2-handle has an ascending 3-ball which meets  $M_{1/2}$  in a smoothly imbedded 2-sphere; call these  $S_{0,i}, i = 1, \dots, n$ . Similarly each 3-handle descends to meet  $M_{1/2}$  in  $S_{1,i}, i = 1, \dots, n$ . We can assume, perhaps after some handle slides, that the boundary map from 3-chains (generated by the 3-handles) to 2-chains (generated by 2-handles) is given by the identity matrix, or, equivalently, that algebraically  $S_{0,i} \cap S_{1,j} = \delta_{ij}$ .

3. Choose a base point  $*$  in  $M_{1/2}$  minus all the spheres  $S_{k,i}$ . Choose  $2n$  arcs in general position which connect  $*$  to basepoints  $*_{k,i}$  in the  $S_{k,i}$ . A regular neighborhood of these arcs will be our 0-handle in a forthcoming handlebody structure on  $M_{1/2}$ . In each  $S_{k,i}$ ,

run disjoint arcs from  $*_{k,i}$  to each point of intersection of  $S_{k,i}$  with some  $S_{k',j}$  (with  $k \neq k'$ ). These arcs come in pairs, from  $*_{k,i}$  to a point in  $S_{k,i} \cap S_{k',j}$  to  $*_{k',j}$ , and a regular neighborhood forms a 1-handle attached to the 0-handle. Each  $S_{k,i}$ , minus the regular neighborhood of the tree of arcs in it, gives a 2-handle,  $H_{k,i}$ , which is added to the 0-handle and 1-handles. Figure 2 shows how the  $H_{k,i}$ 's behave with respect to the 1-handles; note that each  $H_{k,i}$  is attached to an unknot and the  $H_{0,i}$ 's and the  $H_{1,i}$ 's are each attached to unlinks of  $n$  components. Note that we can assume that the  $H_{0,i}$  do not go over any 1-handles, whereas the  $H_{1,i}$  go over and back so that, if the one handles correspond to generators  $x_1, x_2, \dots, x_r$  of the fundamental group  $\pi_1(M_{1/2})$ , then the  $H_{1,i}$  give relators equal to a product of  $x_i \bar{x}_i$ 's and  $\bar{x}_i x_i$ 's,  $i \in 1, \dots, r$ .

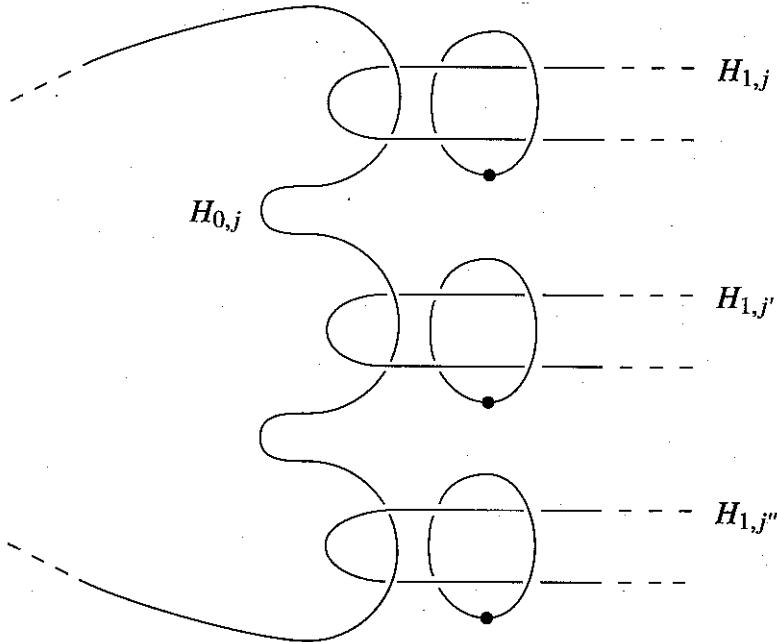


Figure 2.

So far we have chosen a 0-handle and some 1 and 2-handles in  $M_{1/2}$ . Extend this handlebody to a handlebody structure on all of  $M_{1/2}$  (it may have more 1-handles (still indexed by  $1 \dots r$ ) as well as 2 and 3-handles, but extra 0- and 4-handles may be avoided). Since  $M_{1/2}$  is 1-connected and the  $H_{k,i}$  give trivial relators, it follows that the other 2-handles  $H_l, l = 1, \dots, s$  must homotopically kill the 1-handles, where we assume that the attaching circle of each  $H_l$  has a base point  $*_l$  which has been connected by an arc to  $*$ .

Note that when we slide  $H_l$  over  $H_m$ , along an arc  $\lambda$  joining  $*_l$  to  $*_m$  then we replace the relator  $r_l$  by the relator  $r_l \lambda r_m^{\pm 1} \bar{\lambda}$ ;  $\lambda$  can be chosen to be trivial if necessary.

It follows from elementary combinatorial group theory that we can slide 2-handles over 2-handles so as to end up with the  $H_l, l = 1, \dots, r$ , exactly killing the generators  $x_1, \dots, x_r$ ; that is,  $H_i = x_i w_i$  where  $w_i$  cancels away to 1 using only the relations  $x_j \bar{x}_j = 1 = \bar{x}_j x_j$ . During this process, it may have been necessary to add cancelling pairs of 2- and 3-handles, so as to slide a new 2-handle over some  $H_l$  which is about to be altered by sliding over another handle; the new 2-handle preserves the relator  $r_l$  for later use in the sequence of Tietze moves which reduces the original presentation to the *trivial* one. These new 2-3 pairs may be necessary to avoid the difficulties inherent in the Andrews-Curtis Conjecture [2], [6] Problem 5.2.

Let  $B_{1/2}$  be the contractible manifold formed by the 0-handle, all the 1-handles, and the 2-handles  $H_l, l = 1, \dots, r$ . Let  $A_{1/2}$  be  $B_{1/2}$  union the 2-handles  $H_{k,i}, k = 0, 1, i = 1, \dots, n$ . Then  $A$  will be  $A_{1/2}$  (thickened by crossing with  $[1/2 - \epsilon, 1/2 + \epsilon]$ ) together with the 3-handles added below to the  $S_{0,i}$ 's and above to the  $S_{1,i}$ 's.

Clearly  $A$  is contractible (since  $B_{1/2}$  is contractible and the 3-handles cancel the  $H_{k,i}$ ). Since  $A$  contains the 2 and 3-handles of the h-cobordism  $M$ , it follows that  $M - \text{int}A$  is a product h-cobordism. This finishes the proof of the Theorem.

#### 4. Proof of the Addenda:

(B), (C) and (D) are easiest to prove so we start there.

$A$  is diffeomorphic to  $B_{1/2} \times I$  because the 3-handles added to  $A_{1/2}$  geometrically cancel the 2-handles  $H_{k,i}, k = 0, 1, i = 1, \dots, n$ . Furthermore  $B_{1/2} \times I$  is diffeomorphic to  $B^5$  because homotopic circles in a 4-manifold are isotopic, so the attaching maps of the  $H_l$ 's can be isotoped to geometrically cancel the 1-handles, leaving only the 0-handle of  $B_{1/2} \times I$ . This proves (B).

$A_0$  is contractible because  $A$  is, but we need to also know that each 1-handle of  $A_0$  is homotopically cancelled by a 2-handle.  $A_0$  is  $A_{1/2}$  but with a dot on each attaching circle of the  $H_{0,i}$ 's. These dotted circles give  $n$  new generators,  $y_i, i = 1, \dots, n$ , to the presentation for  $\pi_1(B_{1/2})$ , and the  $H_{1,i}, i = 1, \dots, n$ , are  $n$  new relators,  $s_i, i = 1, \dots, n$ . At this point we need to go back and make a careful choice of the arcs in each  $S_{1,i}$  which join  $*_{1,i}$  to the points of intersection of  $S_{1,i}$  with the spheres  $S_{0,j}, j = 1, \dots, n$ . We first run arcs from  $*_{1,1}$  to all points of intersection with  $S_{0,1}$ , then with  $S_{0,2}$ , then  $S_{0,3}$ , and so on to  $S_{0,n}$ . This is easy to do because trees do not separate points in dimension 2. With this choice of arcs, it follows that the attaching circle of  $H_{1,i}$  reads off the word  $w_1 w_2 \dots w_n$  where  $w_j$  is a word in the  $y_j$  and  $\bar{y}_j$  with exponent sum zero if  $j \neq i$  and exponent sum one if  $j = i$ .

Thus the 2-handles  $H_l, l = 1, \dots, r$ , kill  $x_1, \dots, x_r$  and then the 2-handles  $H_{1,i}, i = 1, \dots, n$ , kill the generators  $y_1, \dots, y_n$ . Therefore,  $A_0 \times I$  is diffeomorphic to  $B^5$ , because homotopy implies isotopy for 1-manifolds in 4-manifolds, so the 2-handles geometrically cancel the 1-handles since they do so homotopically. Similarly  $A_1 \times I$  is  $B^5$ . This finishes the proof of Addenda (C).

$\pi_1(L_3)$ , i.e.

$$r'_l \prod_j [a_{l,j}, b_{l,j}]^{c_{l,j}}, \quad l = r + 1, \dots, s,$$

form a set of elements of  $\pi_1(L_3)$  which normally generate it. If we had 2-handles  $H_{l,j}$  whose cocores represented each of the  $b_{l,j}$ , then we could replace  $r'_l$  by  $r'_l \prod_j [a_{l,j}, b_{l,j}]^{c_{l,j}}$  by sliding  $H_{l,j}$  over  $H_l$  and back using arcs  $\lambda_{l,j}$  and  $\mu_{l,j}$  where  $\mu'_{l,j} \bar{\lambda}'_{l,j} = a_{l,j}$  and  $\bar{\mu}'_{l,j} = c_{l,j}$ , and so that each arc is trivial in  $\pi_1(L_1)$  so that the core of  $H_{l,j}$  does not change its homotopy type. Having done this replacement, the cocores of the new  $H_l$ ,  $l = r + 1, \dots, s$ , would kill  $\pi_1(L_3)$ .

So it suffices to find the 2-handles  $H_{l,j}$ . Suppose there are  $m$  of the  $b_{l,j}$ . Then we introduce  $m$  cancelling 1-,2-handle pairs into the handlebody structure on  $M_{1/2}$  and include these pairs in  $A_{1/2}$ . Each  $b_{l,j}$  is a product of conjugates of the relators  $r'_l$ ,  $l = 1, \dots, s$ , so if we slide the corresponding 2-handles  $H_l$  or  $H_{k,i}$  over  $H_{l,j}$ , then the cocore of  $H_{l,j}$  slides over the cocores and ends up representing  $b_{l,j}$ . Of course, the core of  $H_l$  still kills its original 1-handle, and sliding  $H_{k,i}$  merely changes the isotopy class of the 2-sphere  $S_{k,i}$ . (This step is essentially nothing but the observation that one can always add a cancelling pair of 2-,3-handles where the 2-handle represents any desired word in the 1-handles.)

It may be useful to summarize here the whole construction. In  $M_{1/2}$ , choose a base point,  $*$ , hence a 0-handle, and then  $r$  1-handles corresponding to each point of intersection between the ascending and descending 2-spheres. Each of these 2-spheres then provides a 2-handle  $H_{k,i}$ . Extend this handle structure to  $M_{1/2}$ . Slide 2-handles to get  $r$  2-handles  $H_l$  which homotopically cancel the 1-handles (stabilization by cancelling pairs of 2-, 3-handles to avoid Andrews-Curtis issues may have been necessary). Add some spare pairs of cancelling 1-,2-handles for later use. Inverting  $M_{1/2}$  so that 1-handles become 3-handles, etc., we slide the spare 2-handles over the other 2-handles so that they will represent certain words, namely the  $b_{l,j}$ . Then we slide the 2-handles  $H'_l$ ,  $l = r + 1 \dots s$ , over the spare 2-handles and back so as to create relators which kill  $\pi_1(L_3)$ .

Now  $B_{1/2}$  will consist of all of the 1-handles and all their homotopically cancelling 2-handles, so that  $B_{1/2}$  is contractible.  $A_{1/2}$  is  $B_{1/2}$  union the  $H_{k,i}$ 's, as before. Finally  $C_{1/2}$  is the 3-handles union the final version of the  $H_l$ ,  $l = 1, \dots, s$ . Both  $A_{1/2}$  and  $C_{1/2}$  are simply connected.

We use this new  $A_{1/2}$  and proceed to prove Addenda (B), (C) and (D) as before (it is easy to check in proving (D) that the complement remains simply connected). This completes the proof of the Theorem and all its Addenda.

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