MAZUR MANIFOLDS

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1. INTRODUCTION

In [10] Mazur constructed a contractible 4-manifold whose boundary is a homology 3-sphere not equal to S^3. In this paper we investigate some generalized Mazur manifolds $W^{\pm}(\ell,k)$ gotten by adding a 2-handle to $S^1 \times B^3$ in certain ways. Consider the knots $K^{\pm}$ in $S^1 \times B^2 \subset S^1 \times S^2 = \partial(S^1 \times B^3)$ drawn below in Figure 1.

There are ℓ full twists (right-handed as drawn if $\ell > 0$, left-handed if $\ell < 0$) in $K^{\pm}$. The 0-framing on the normal bundle to $K^{\pm}$ is the one derived from the normal vector field which is tangent to a Seifert surface between $K^{\pm}$ and the curve $\gamma_{\pm}(= S^1 \times q$ where $S^1 \times q \cap K^{\pm} = \emptyset)$. Since $\pi_1(SO(2))$ acts on the normal bundle to $K^{\pm}$ in the obvious way, twisting k times, k determines a trivialization of the normal disk bundle which we use to attach a 2-handle to $S^1 \times B^3$, getting $W^{\pm}(\ell,k)$. Mazur's example ([10]) was $W^{-}(0,3) \approx W^{+}(0,0)$, (see section 2 for the diffeomorphism).

We consider the question: is $\gamma_{\pm}$ homotopically slice? That is, does $\gamma_{\pm}$ bound a *smoothly* imbedded disk in some contractible 4-manifold X^4 with $\partial X^4 = \partial W^{\pm}(\ell,k)$?

THEOREM 1. *γ_- is homotopically slice if and only if*

$$(\ell,k) = (0,0), \quad (4,1) \quad \text{or} \quad (2,k).$$

THEOREM 1$'$. *γ_+ is homotopically slice if and only if*

$$(\ell,k) = (2,1), \quad (-2,0) \quad \text{or} \quad (0,k).$$

Theorem 1$'$ follows from Theorem 1 because there is a diffeomorphism between $\partial W^{-}(\ell,k)$ and $\partial W^{+}(-\ell+2, -k+1)$ which takes γ_- to γ_+ (see Proposition 1, section 2).

Zeeman [13, page 357] suggested that no essential knot in the boundary of a contractible manifold is slice. Somewhat the opposite has turned out to be true (see [9] for some examples), for R. Fenn [3] showed that any circle in the boundary of a contractible manifold with a 2-dimensional spine is homotopic to one which is slice. However, some special cases are still interesting. It has been known for some time that γ_+ is slice in $W^{+}(0,k)$, for all k (the slice is drawn in section 5). However the same method cannot work for γ_- in $W^{-}(0,0)$ (see [9]). Considerable effort has not produced a slice. But as is known (section 5), γ_- is slice in some other contractible manifold W. W is h-cobordant to $W^{-}(0,0)$ (any contractible

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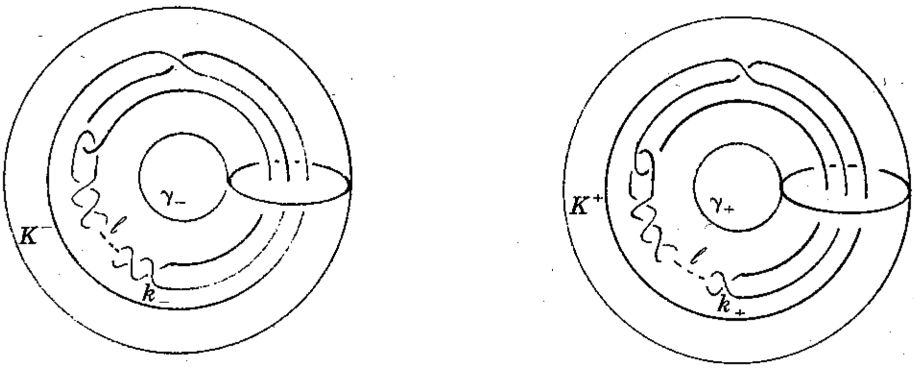

$W^-(\ell, k)$ $W^+(\ell, k)$

Figure 1

manifolds W_1 and W_2 with the same boundary are h-cobordant because $W_1 \underset{\partial}{\cup} -W_2$ is a homotopy 4-sphere which then bounds a homotopy 5-ball). So either the h-cobordism fails to be trivial or there is a slice no one has found.

Zeeman also asks [13] whether curves such as γ_- in $\partial W^-(0,0)$ bound PL disks. Our methods shed no light on this difficult question for we need smooth, hence locally flat disks.

In section 7, we show that three of the Mazur boundaries are Brieskorn homology spheres, which thus bound contractible manifolds. Let

$$\Sigma(a,b,c) = \{(x,y,z) \in \mathbb{C}^3 : x^a + y^b + z^c = 0\} \cap S^5$$

(see [11]).

THEOREM 2 (1) $\Sigma(2,5,7) \approx \partial W^+(0,0)$

(2) $\Sigma(3,4,5) \approx \partial W^+(-1,0)$

(3) $\Sigma(2,3,13) \approx \partial W^+(1,0)$.

A. J. Casson and J. L. Harer [2] have since shown that many Brieskorn homology spheres bound contractible manifolds built using one 1-handle and one 2-handle.

Section 2 contains equivalent descriptions of Mazur manifolds; in sections 3, 4 we calculate various invariants to show that $\gamma_\pm$ is not slice; section 5 contain the proof of Theorems 1, 1′; and section 6 has the proof of Theorem 2. In the remainder of this section some notation is fixed.

The reader should be familiar with the language of [7]. In particular, a framed link in S^3 provides attaching maps for adding 2-handles to B^4, and thereby determines a simply connected, smooth 4-manifold with boundary. If we add a 1-handle to B^4 we get $S^1 \times B^3$. The attaching map of a 2-handle can then be drawn as in the picture above for $W^\pm(\ell, k)$. However it is more convenient to draw the "linking circle" of the 1-handle, that is, $* \times \partial B^2 \to S^1 \times S^2 \to S^1 \times B^3$. We put a dot on this circle to indicate a 1-handle. The picture for $W^-(\ell, k)$ becomes:

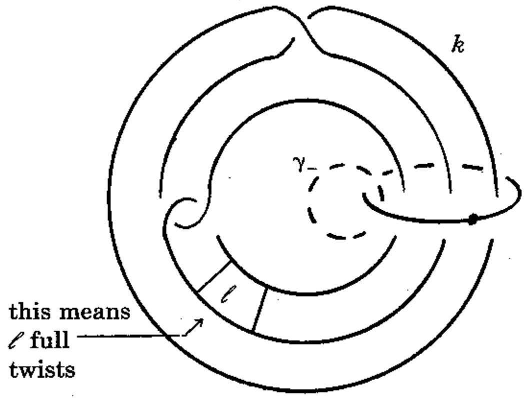

Figure 2

This notation is convenient because we frequently want to surger the circle determined by the 1-handle. This amounts to replacing the 1-handle by a 2-handle which is attached to the same circle with framing 0 (and the dot is removed). Conversely, any unknotted, unlinked collection of 0-framed circles determine disjoint, smooth 2-spheres with trivial normal bundles. Surgery on these 2-spheres amounts to replacing the 2-handles by 1-handles which are denoted by the same circles with dots. These surgeries, or exchanges of handles, obviously change the 4-manifold, but never its boundary. This is one tool in obtaining different framed link descriptions of the same 3-manifold boundary.

The phrases "blow up" and "blow down" refer to the operations in Propositions 1A and 1B of [7]; a $\mathbb{C}P^2$ or $\overline{\mathbb{C}P}^2$ is being added to (or subtracted from) a 4-manifold, and 2-handles are slid over (off) the new 2-handle. Throughout the paper, $\approx$ means diffeomorphism, and $\overset{\partial}{\approx}$ means that the boundaries of the corresponding manifolds are diffeomorphic. Twists means full twists; half twists are indicated when they occur.

If a 2-handle goes geometrically once over a 1-handle, then the pair may be cancelled. In a framed link diagram, this is done in two steps: (1) if any other two handles go over the 1-handle (*i.e.*, go through the dotted circle) then they must be slid over the cancelling 2-handle, (2) the cancelling handles are erased. Note that this process may be reversed, introducing a dotted circle which links only the cancelling 2-handle which is allowed to go *anywhere* else.

2. EQUIVALENT DESCRIPTIONS OF MAZUR MANIFOLDS

PROPOSITION 1. (1) $W^{\pm}(\ell, k) \approx W^{\pm}(\ell + 1, k - 1)$

(2) $W^{-}(\ell, k) \approx W^{+}(-\ell + 2, -k + 1)$

(3) $W^{-}(\ell, k+1) \# \mathbb{C}P^2 \approx K^{+1}$ (*interior connected sum*)

(4) $W^{+}(\ell, k+1) \# (-\mathbb{C}P^2) \approx K^{-1}$

where

$K =$

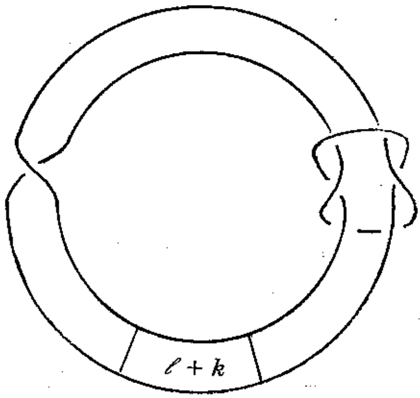

Figure 3

Proof of (1). $W^-(\ell, k)$ is diffeomorphic to the first diagram in Figure 4, and after 3 handle slides, the last diagram is diffeomorphic to $W^-(\ell + 1, k - 1)$.

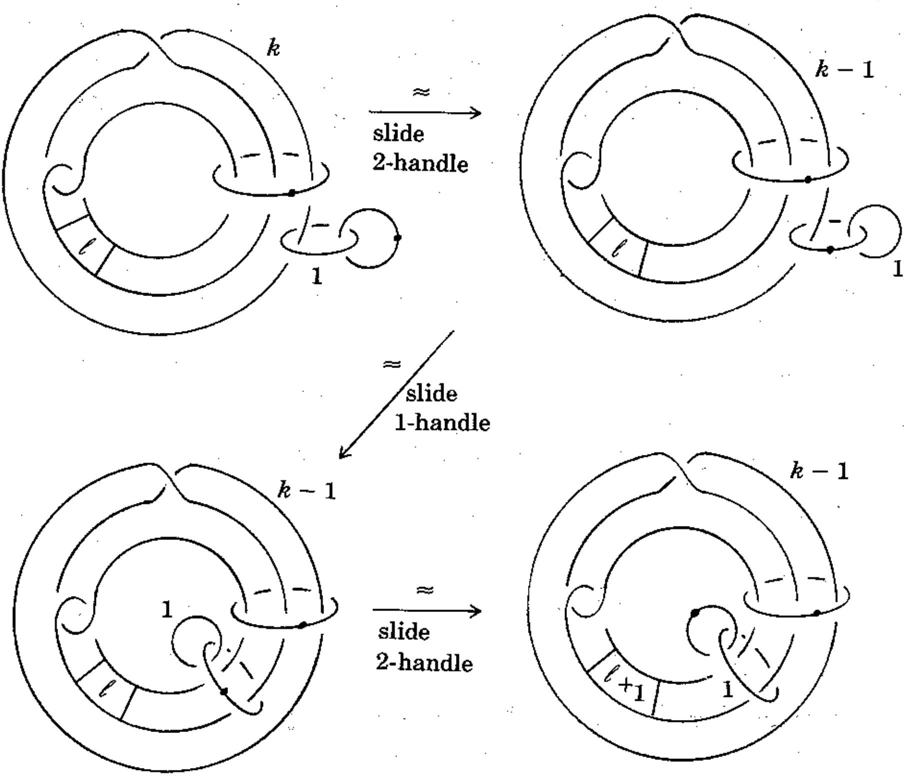

Figure 4

Proof of (2). $W^-(\ell,k)$ is diffeomorphic, after sliding one 1-handle over the other, to the first diagram in Figure 5. After cancelling the 1-2-pair in the last diagram and changing orientation, we get $W^-(-\ell+2,-k+1)$.

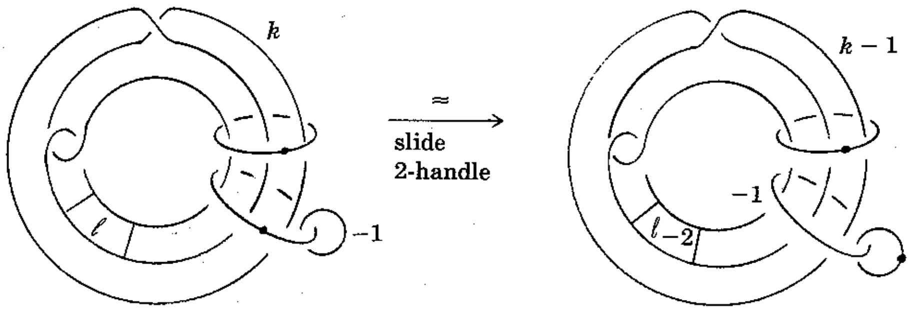

Figure 5

Proof of (3). We proceed through the 4 diagrams in Figure 6. In the last diagram, cancel the one and 2-handles, and redraw the remaining 2-handle to finish the proof.

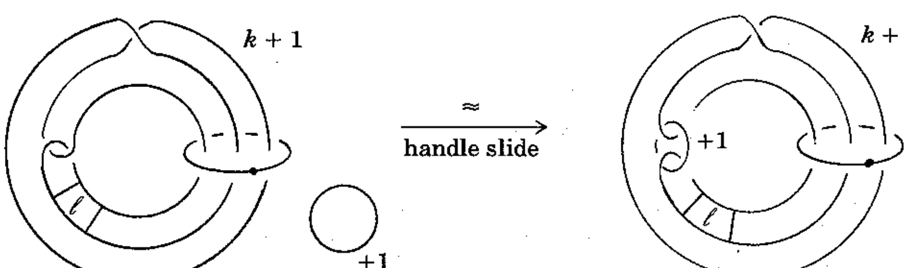

Figure 6

The proof of (4) is simi.ar and we omit it.

Remark 1. One can easily show that $W^-(\ell,k+1)\bigcup_{\partial K^{+1}}(-K^{+1})\approx \mathbb{C}P^2$ the same holds for $W^+(\ell,k+1)$.

Remark 2. The diffeomorphism (2) takes γ_- to γ_+, and reverses orientation.

Remark 3. (1) implies that: $W^{\pm}(\ell,k)\approx W^{\pm}(\ell',k')$ when $\ell+k=\ell'+k'$. Unfortunately this diffeomorphism does not fix $\gamma_\pm$.

Remark 4. By (1) and (2), $W^{\pm}(\ell,k)$ is diffeomorphic (perhaps reversing orientation) to one of $W^+(0,k)$. Laudenbach and Eaton have shown that $\partial W^+(0,k)$ is never simply connected.

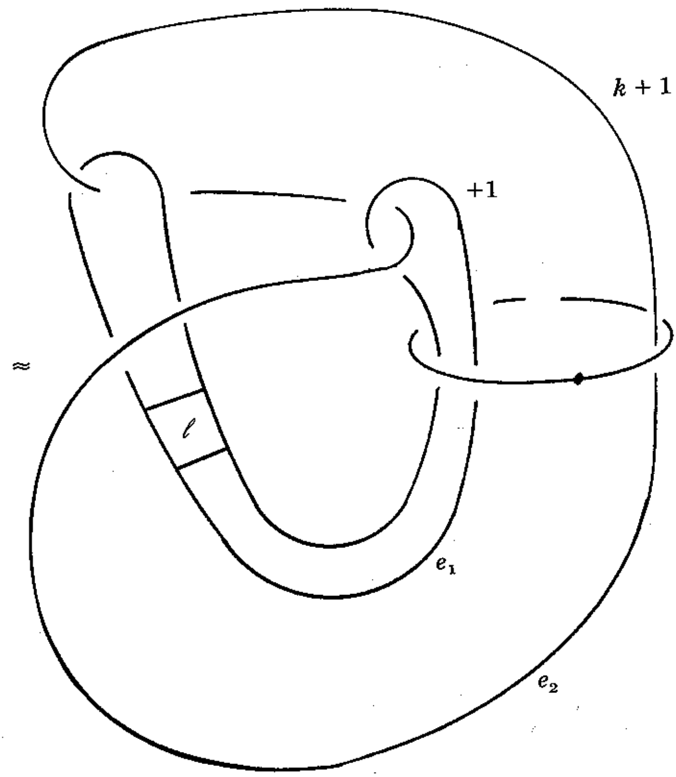

Figure 6 continued

Remark 5. It is interesting to note that the diffeomorphisms (3) and (4) carry the $\pm\mathbb{C}P^1$ in $\pm\mathbb{C}P^2$ to the obvious 2-sphere in $K^{\pm 1}$ (namely the slice disc K bounds in B^4 union the core of the 2-handle).

3. CALCULATION OF ALGEBRAIC INVARIANTS OF γ

To calculate the algebraic invariants of γ we have to find a Seifert surface for γ. For $\gamma \subset W^-(\ell,-r)$, by rotating $S^1 \times B^3$ by a diffeomorphism r times and extending it to a diffeomorphism over attaching 2-handles, we get the diffeomorphism in Figure 7. We have changed the interior of $W^-(\ell,k)$; now γ lies on the boundary of a different Mazur manifold (with the boundary $\approx \partial W^-(\ell,k)$). To find a Seifert surface for γ we first draw a surface between γ and the boundary of the core 2-disc of the 2-handle in $S^1 \times S^2$ (*i.e.,* a copy of the knot δ pushed off with zero framing; see Figure 8). We can draw this surface in $S^1 \times S^2$ because "δ" is a 0-framed knot. Then we cap off this surface with D^2 over the 2-handle in $\partial W^-(\ell,k)$. Hence, in Figure 8 we only see the part of the Seifert surface which is in $S^1 \times S^2$ (the Seifert surface $- D^2$), but this is enough for calculating the Seifert form of γ. The Seifert matrix of γ is $L_1 = (\lambda_{ij})$, where

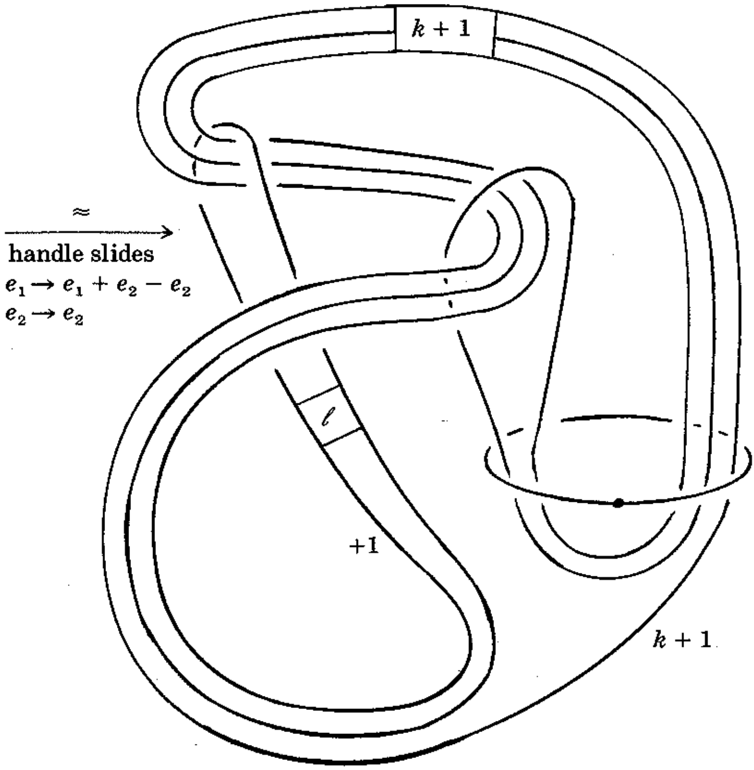

Figure 6 continued

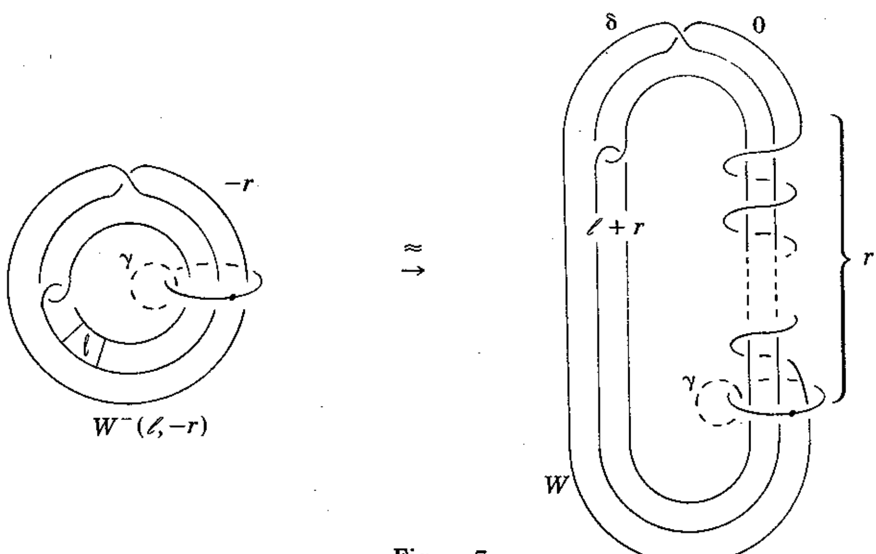

Figure 7

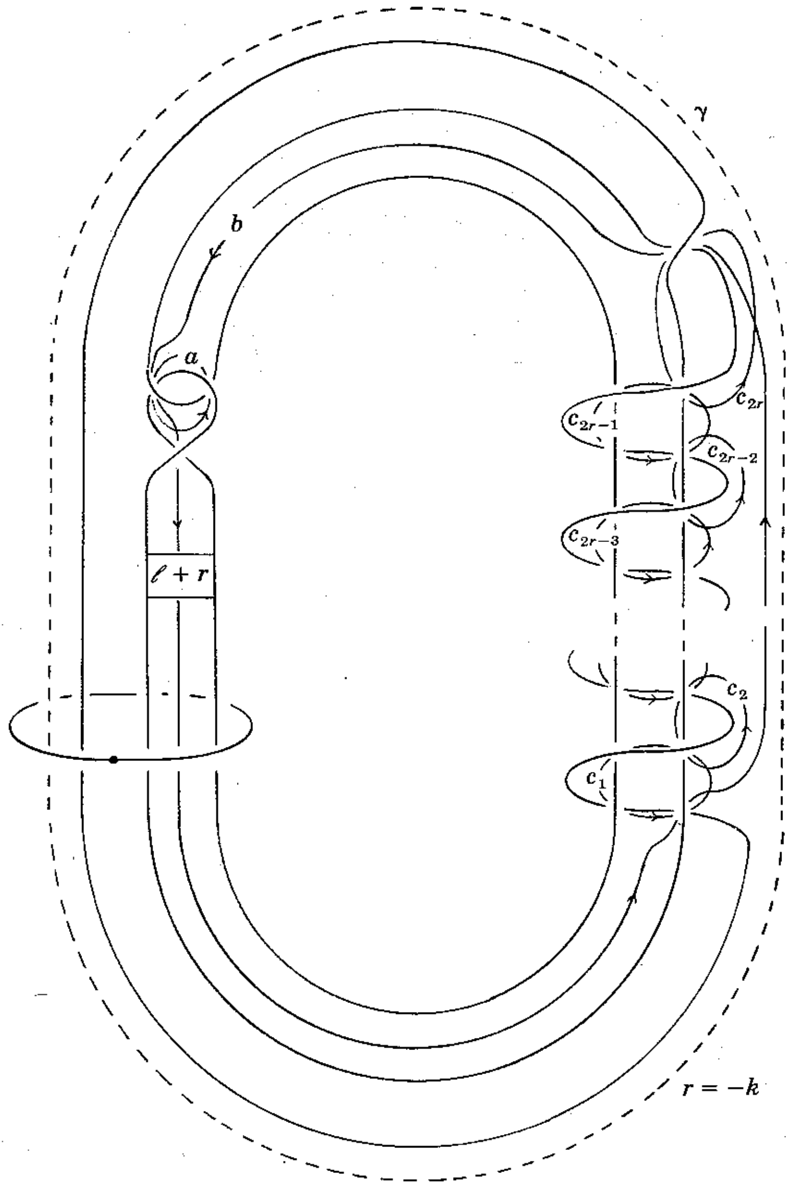

Figure 8

$$\lambda_{ij} = \text{linking number } (\lambda_i, i_* \lambda_j),$$

and where the λ_i's are curves generating the homology of the Seifert surface, and i is a vector field normal to the Seifert surface. We get the matrix below.

$$L_1 = \left(\begin{array}{cc|cccccccc} -1 & 0 & & & & & & & & \\ -1 & \ell + r - 1 & -1 & & & & & & & \\ \hline & & 0 & 1 & & & & & & \\ & & & -1 & 1 & & & & & \\ & & & & 0 & 1 & & & & \\ & & & & & -1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & & 0 & 1 \\ & -1 & & & & & & & & -1 \end{array}\right) \quad \Bigg\} \; 2r$$

In the case $\gamma \subset \partial W^-(\ell, k)$ and $k > 0$, by first changing the orientation of $W^-(\ell, k)$ and proceeding as in the previous case, we get a $2k \times 2k$ Seifert matrix L_2 which differs in form from L_1 only in the upper left-hand block which is $\begin{pmatrix} 1 & 1 \\ 0 & -\ell + k \end{pmatrix}$ for L_2.

(a) *Calculation of the p-signatures of* γ. When $\gamma \subset \partial W^-(\ell, k)$ and $k \leq 0$,

$$\sigma_p(\gamma) = \text{Signature } \{(1/2)(1 - \bar{\omega}_p)(L_1 - \omega_p L_1^T)\}$$

(see [12]) where $\omega_p = \exp(2m\pi i/(2m+1))$ for an odd prime $p = 2m+1$, and $\omega_2 = -1$.

When $\gamma \subset \partial W^-(\ell, k)$ and $k > 0$,

$$\sigma_p(\gamma) = -\text{ Signature } \{(1/2)(1 - \bar{\omega}_p)(L_2 - \omega_p L_2^T)\}.$$

The negative sign is because we have changed the orientation of $W^-(\ell, k)$ in the first step of our modification. For $\gamma \subset \partial W^-(\ell, k)$, we get

$$\sigma_p(\gamma) = \begin{cases} -1 + \operatorname{Sgn}\left(\ell - 1 + \dfrac{2\operatorname{Re}\omega^{-k}(1-\omega) + 1}{2\operatorname{Re}(1-\omega)}\right) & \text{for } k \leq 0 \\[2ex] -1 + \operatorname{Sgn}\left(\ell - 1 + \dfrac{-2\operatorname{Re}\omega^{k-1}(1-\omega) + 1}{2\operatorname{Re}(1-\omega)}\right) & \text{for } k > 0 \end{cases}$$

where $\omega = \omega_p$ and $\operatorname{Sgn}(a) = a/|a|$ when $a \neq 0$, or equals 0 when $a = 0$.

(b) *Calculation of the Alexander polynomial of* γ. When $\gamma \subset \partial W^-(\ell,k)$, $k = -r \leq 0$, the Alexander polynomial of γ is (see [8]);

$$\begin{aligned}\Delta_{L_1}(t) &= \text{determinant}\,(tL_1 - L_1^T)\\ &= t^{r+1}[-t^{r+1} + t^r - (\ell-1)t + (2\ell-1) - (\ell-1)t^{-1} + t^{-r} - t^{-r-1}].\end{aligned}$$

Similarly, when $\gamma \subset \partial W^-(\ell,k)$, $k > 0$, then

$$\begin{aligned}\Delta_{L_2}(t) &= \text{determinant}\,(tL_2 - L_2^T)\\ &= t^k[t^k - t^{k-1} - (\ell-1)t + (2\ell-1) - (\ell-1)t^{-1} - t^{-(k-1)} + t^{-k}].\end{aligned}$$

4. ESTIMATING THE CASSON-GORDON INVARIANT OF γ

Let $\gamma \subset \partial W^-(\ell,k)$, $k = -r \leq 0$; also assume $\ell = b(b+1) + 1 + (-1)^{r+1}$ where b is an integer (we restrict ourselves to this range because this is when the Casson-Gordon invariant of γ is non-trivial as will become clear later on). From section 3 we can assume that γ lies in the boundary of W (see section 3).

Claim. $\partial(\gamma^\circ) \approx \partial(K^\circ_{\ell,r})$, where γ° is the 4-manifold obtained by attaching a 2-handle to W^4 using the 0-framing on $\gamma \subset \partial W$, and $K_{\ell,r}$ is the knot "δ" of section 3 (we rename δ for a technical reason).

Proof. $\gamma^\circ = (S^1 \times B^3 \cup \text{2-handle along } \delta) \cup (\text{2-handle along } \gamma)$

$\approx (S^1 \times B^3 \cup \text{2-handle along } \gamma) \cup (\text{2-handle along } \delta)$

$\approx B^4 \cup \text{2-handle along } K_{\ell,r} = K^\circ_{\ell,r}$.

According to [1] γ has the same Casson-Gordon invariant as $K_{\ell,r}$. First recall the definition of Casson-Gordon invariant in the following special case (see [1]): let M^3 be a closed 3-manifold such that

(i) $M^3 = \partial W^4$, W^4 is compact,

(ii) $H_1(M^3) = \mathbb{Z}_{m^2} \xrightarrow{i_*} H_1(W) = \mathbb{Z}_m$ is onto where i is the inclusion.

Let $\chi\colon \mathbb{Z}_m \to \mathbb{Z}_q$ be an epimorphism. Let $\tilde{W}$ be the q-fold covering of W induced by χ. Then the chain complex $C_*(\tilde{W})$ is a $\mathbb{Z}[\mathbb{Z}_q]$ − module. Let k be the cyclotomic field $\mathbb{Q}(\mathbb{Z}_q) \subset \mathbb{C}$ ($\mathbb{Z}_q$ is identified with the group of q-th roots of 1); then define

$$H_*(W{:}k) = H_*(C_*(\tilde{W}) \underset{\mathbb{Z}[\mathbb{Z}_q]}{\otimes} k).$$

There is a hermitian pairing: $H_2(W;k) \otimes H_2(W;k) \to k$ defined by

$$\langle x,y\rangle = \sum_{t\in \mathbb{Z}_q} (x,ty)\,t^{-1}$$

where $(,)$ is the ordinary pairing of $H_2(\tilde{W}{:}\mathbb{Z})$. Now define

$$\psi(M^3,\chi) = \text{Signature } H_2(W{:}k) - \text{Signature } H_2(W{:}\mathbb{Z}).$$

Let K be a knot in a homology sphere V^3 which is a boundary of a homology 4-ball. If the 2-fold branched covering space M^3 of V^3 branched along K satisfies (i) and (ii), we define $\psi(K;\chi) = \psi(M^3{:}\chi)$. In [1] it is shown that if such a knot K is slice in a homology ball and q is a prime power then $|\psi(K;\chi)| \le 1$. If m is prime for brevity we denote $\psi(K;\chi)$ by $\psi(K)$.

We are now ready to estimate $\psi(K_{\ell,r},\chi)$. To find the 2-fold branched covering space of $K_{\ell,r}$, we first unknot $K_{\ell,r}$ by blowing up as in Figure 9, and then (Figure

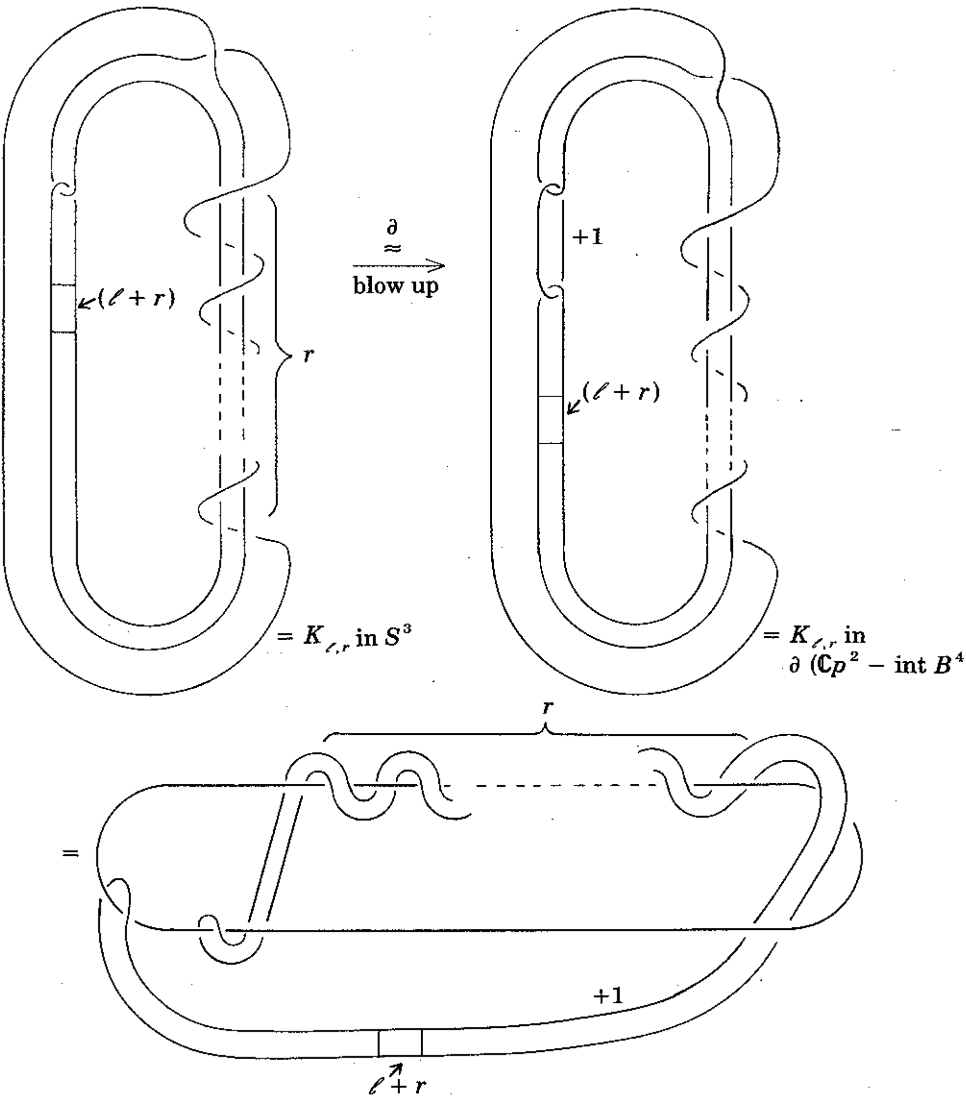

Figure 9

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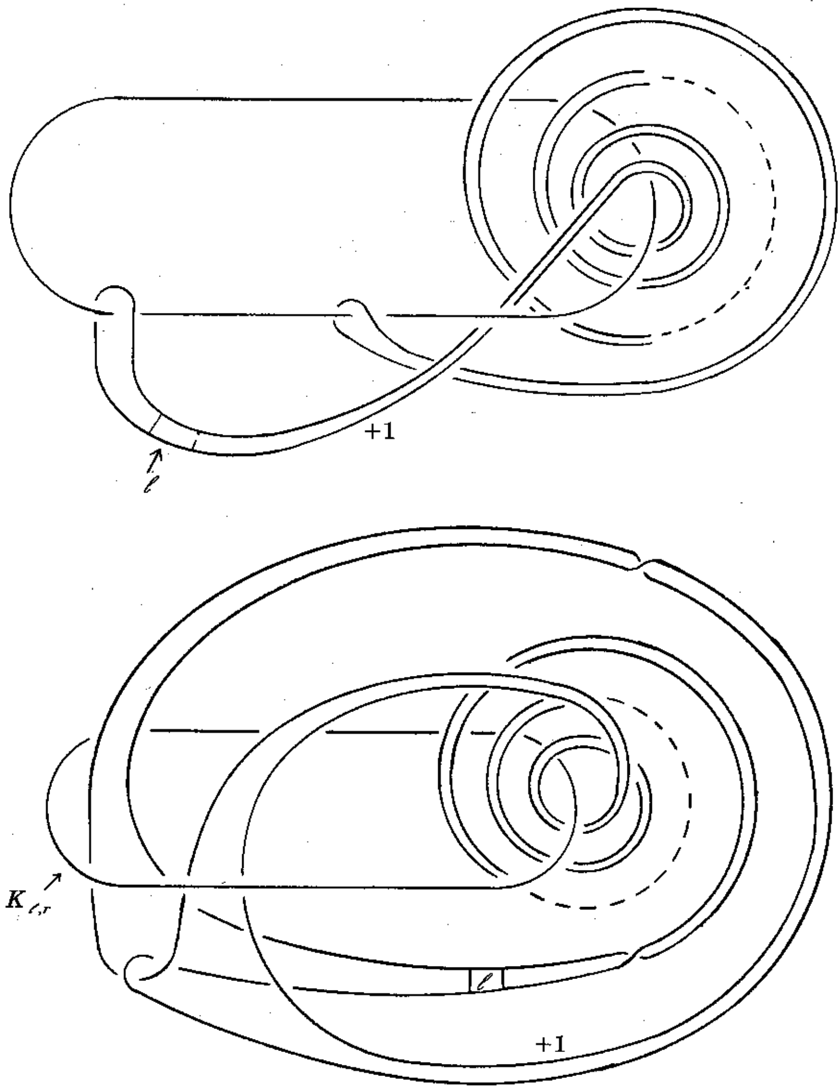

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Figure 9 continued

10) branch along the obvious B^2 with $\partial B^2 = K_{\ell,r}$ to obtain a framed link picture for the double branched cover of $CP^2 - \operatorname{int} B^4$ along B^2. The framings in Figure 10, which are the intersections of the homology classes C_1 and C_2 in the covering space, can be computed in terms of the homology class C generating

$$H_2(CP^2 - \operatorname{int} B^4; \mathbb{Z})$$

by the formula: $C_i \cdot C_1 + C_i \cdot C_2 = C \cdot C$ *for* $i = 1$ or 2.

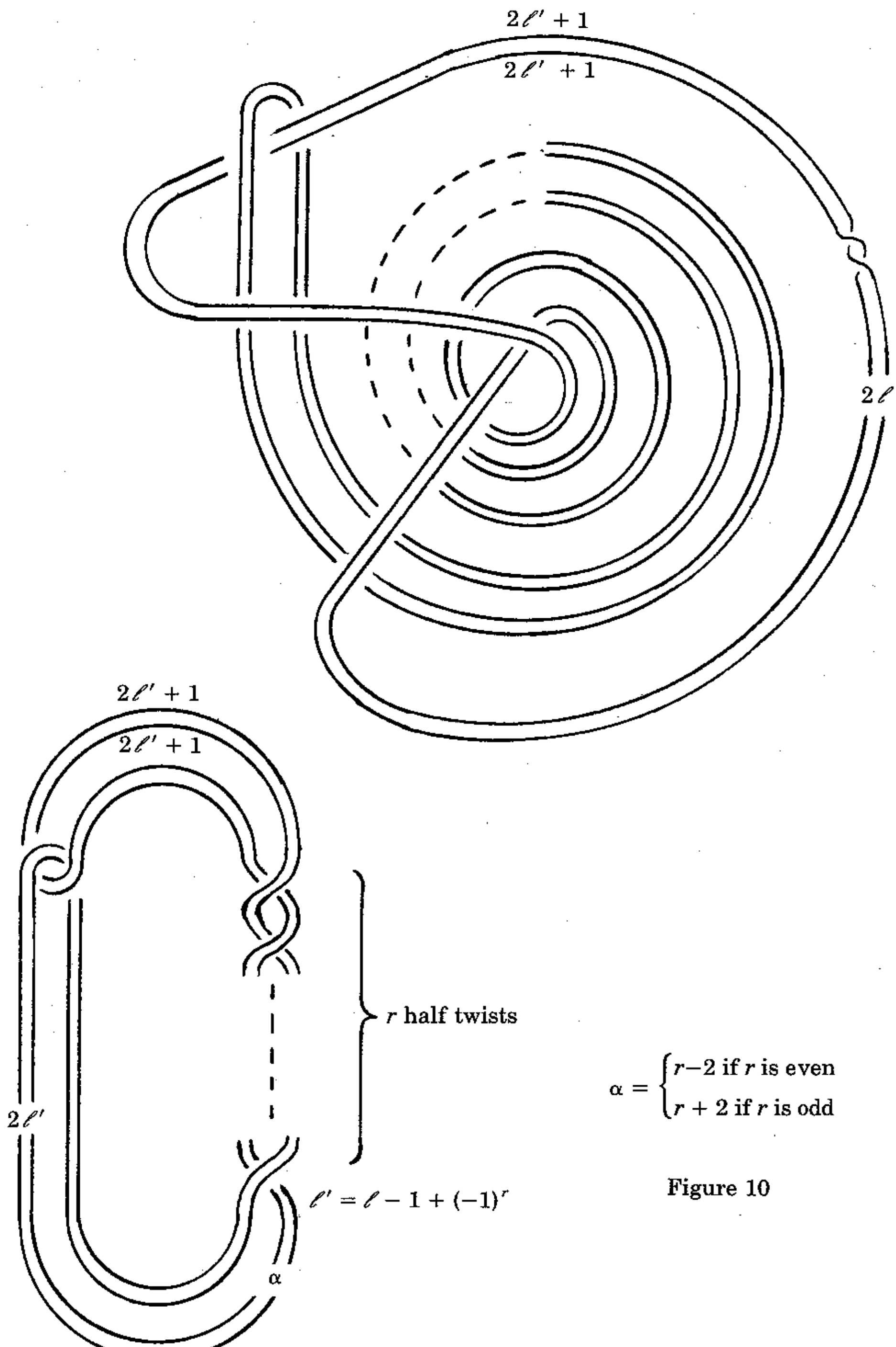
$2\ell' + 1$
$2\ell' + 1$
2ℓ
$2\ell' + 1$
$2\ell' + 1$
r half twists
$2\ell'$
$\ell' = \ell - 1 + (-1)^r$
α
$\alpha = \begin{cases} r-2 \text{ if } r \text{ is even} \\ r+2 \text{ if } r \text{ is odd} \end{cases}$

Figure 10

Therefore the 2-fold branched covering space of $K_{\ell,r}$ can be described by ∂M_1 when r is even, and by ∂M_2 when r is odd, where M_1 and M_2 are given in Figure 11.

Let

$$M = \begin{cases} M_1 \text{ if } r \text{ is even} \\ M_2 \text{ if } r \text{ is odd.} \end{cases}$$

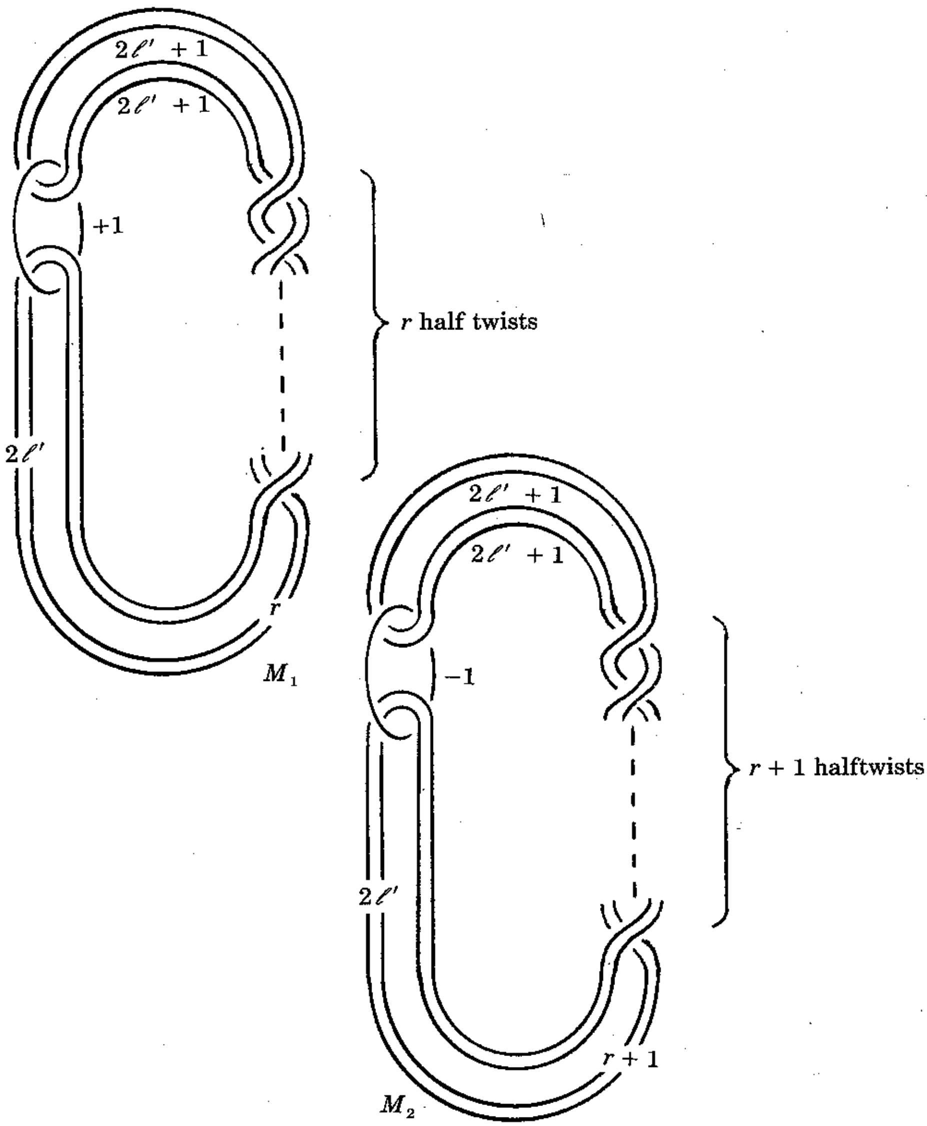

Figure 11

Claim. The 2-fold branched covering space of $K_{\ell,r}$ is obtained by doing a surgery along a knot in the two-fold cyclic branched covering space of the rational knot $\xi_{\ell'}$ in Figure 12.

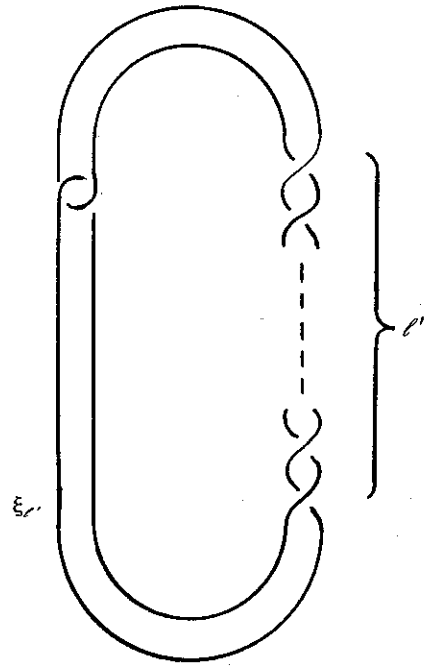

Figure 12

Proof of Claim. Again we take the 2-fold branched covering of $\xi_{\ell'}$ by blowing up to unknot $\xi_{\ell'}$ and then constructing the covering as in Figure 13. Then clearly M is obtained by adding a handle h with ± 1 framing onto the 2-fold cyclic branched covering ∂N^4 of $\xi_{\ell'}$.

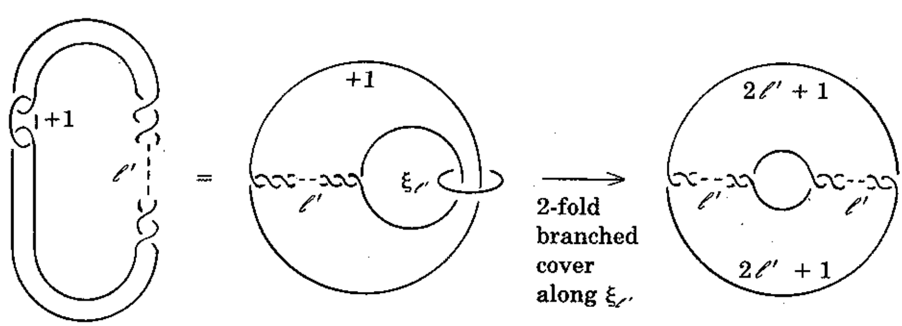

Figure 13

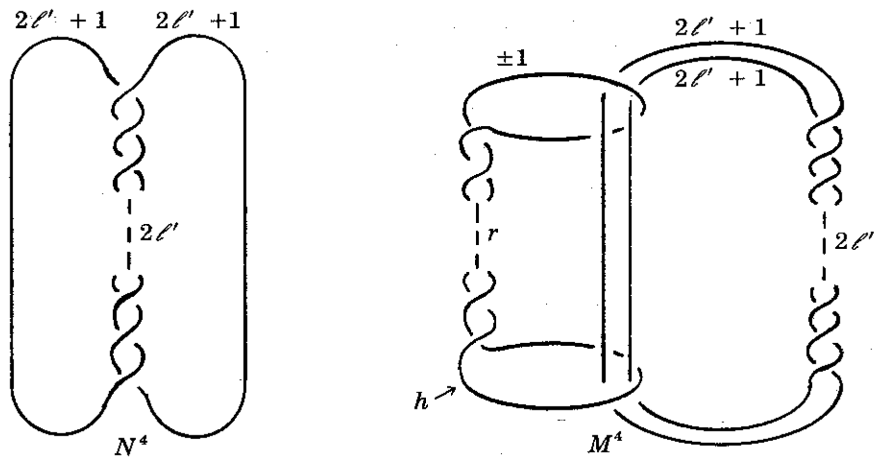

Figure 14

Call $\partial N^4 = L$. We have $M^4 = N^4 \cup h$, and $H_1(\partial M^4) = H_1(L) = \mathbb{Z}_u$ with

$$u = \det\begin{pmatrix} 2\ell' + 1 & 2\ell' \\ 2\ell' & 2\ell' + 1 \end{pmatrix} = 4\ell' + 1 = 4b(b+1) + 1 = (2b+1)^2.$$

Let χ be any epimorphism $\mathbb{Z}_{2b+1} \to \mathbb{Z}_q$ where q is a prime-power dividing $2b + 1$. Let W^4 be a manifold which bounds L and satisfies (i) and (ii). Then $W \cup h$ bounds ∂M and satisfies (i) and (ii). Let $\tilde{W}$ be the q-fold covering of W induced by χ; then $\tilde{W} \cup h_1 \cup \ldots \cup h_q$ is the induced q-fold covering of $W \cup h$, where h_i, $1 \le i \le q$, are the handles covering h.

$$\begin{aligned} H_2(W \cup h; k) &= H_2(C_*(\tilde{W} \cup h_1 \cup \ldots \cup h_q) \underset{\mathbb{Z}[\mathbb{Z}_q]}{\otimes} k) \\ &= H_2(C_*(\tilde{W}) \underset{\mathbb{Z}[\mathbb{Z}_q]}{\otimes} k) \oplus k \end{aligned}$$

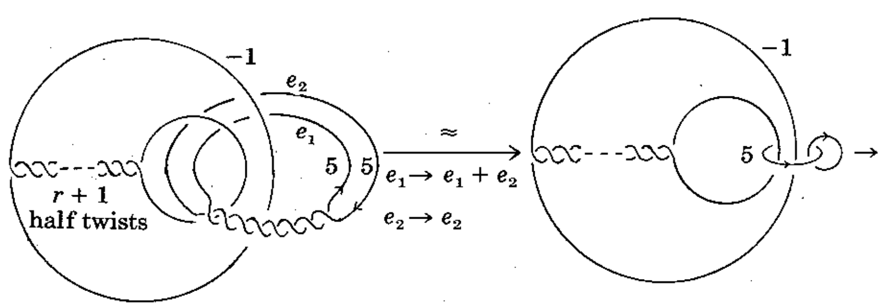

Figure 15

since as a $\mathbb{Z}[\mathbb{Z}_q]$-module we have the relations $h_i = t^{i-1}h_1$ where t is the multiplicative generator of $\mathbb{Z}_q$ and h_1 is a cycle. Therefore:

$$\begin{aligned}\psi(K_{\ell,r},\chi) &= \text{Signature}\, H_2(W \cup h;k) - \text{Signature}\, H_2(W \cup h{:}\mathbb{Z}) \\ &= \text{Signature}\, H_2(W{:}k) \pm 1 - (\text{Signature}\, H_2(W{:}\mathbb{Z}) \pm 1) \\ &= \psi(\xi_{\ell'},\chi) \pm 1 \pm 1 \leqq \psi(\xi_{\ell'},\chi) + 2.\end{aligned}$$

But in [1] $\psi(\xi_{\ell'},\chi)$ is calculated to be -5 for some χ when $\ell' = b(b+1) > 2$. Therefore $|\psi(\gamma,\chi)| \geq 3$ when r is even and $\ell = b(b+1) > 2$, or r is odd and $\ell = b(b+1) + 2 > 4$. If r is odd and $\ell = 4$ we calculate $\psi(\gamma) = \psi(K_{4,r})$ directly from Figure 15. Starting with the 2-fold branched covering of γ, ∂M_2, we do

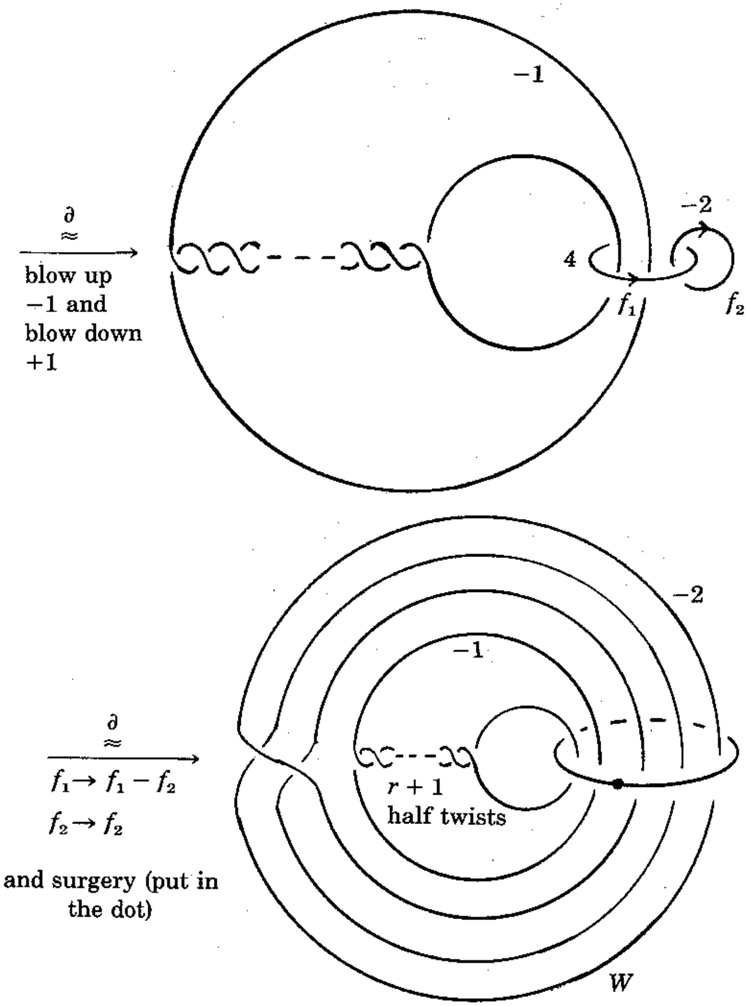

Figure 15 continued

a handle addition to get the second diagram, blow up a minus one to separate the 5 and 2 and then blow down a plus one to get the third diagram; finally a handle addition and surgery defines W. Clearly $\mathbb{Z}_9 = H_1(\partial W) \xrightarrow{i_*} H_1(W) = \mathbb{Z}_3$ is onto, where i is the inclusion $\partial W \subset W$. Choose i_* to be the character χ. We now take the 3-fold covering $\tilde{W}$ of W, drawn in Figure 16. The self-intersections of the 2-spheres α_i's, β_i's are computed as before by the formulas

$$\alpha_i \cdot \alpha_i + \sum_{j \neq i} \alpha_i \cdot \alpha_j = -2, \qquad \beta_i \cdot \beta_i + \sum_{j \neq i} \beta_i \cdot \beta_j = -1 \qquad (i = 1, 2, 3).$$

$H_2(W{:}k)$ is generated by β_1 and

$$\begin{aligned} \langle \beta_1, \beta_1 \rangle &= (\beta_1, \beta_1) + (\beta_1, \beta_2)\, t^2 + (\beta_1, \beta_3)\, t \\ &= r - \left(\frac{r+1}{2}\right) t^2 - \left(\frac{r+1}{2}\right) t = \frac{3r+1}{2} \end{aligned}$$

where $t = e^{2\pi i/3}$, $\beta_i = t^{i-1}\beta_1$. Then

$$\psi(\gamma) = \text{Signature}\left(\frac{3r+1}{2}\right) - \text{Signature}\,(-1) = 2.$$

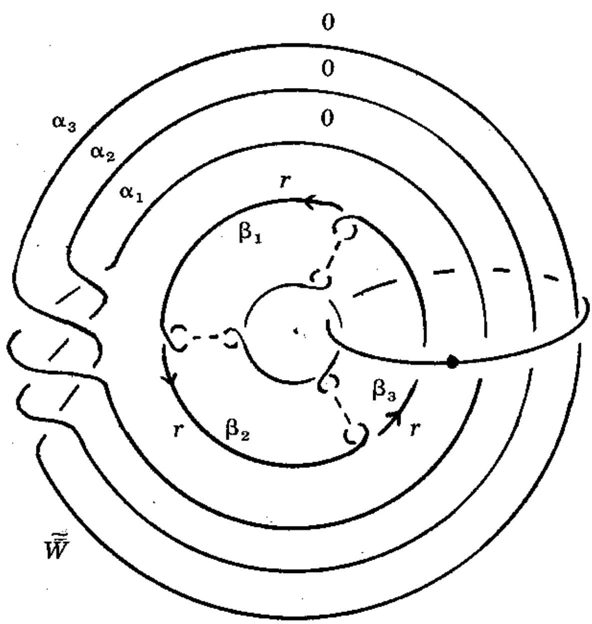

Figure 16

For $\gamma \subset \partial W^-(\ell,k)$, $k > 0$, by a similar calculation we get $|\psi(\gamma)| \geq 3$ when

$$\ell = b(b+1) + 1 + (-1)^{k+1} > 2,$$

and $(\ell,k) \notin \{(4,2n+1) | n \geq 0\}$.

5. PROOF OF THEOREMS 1 AND 1′

We prove, for $\ell \leq 0$ and $k \neq 0$, that γ is not slice by showing $\sigma_p(\gamma) \neq 0$ for some prime p. Say, for example, $k < 0$; then

$$\sigma_p(\gamma) = -1 + \operatorname{Sgn}\left(\ell - 1 + \frac{2\mathrm{Re}\,\omega^{-k}(1-\omega) + 1}{2\mathrm{Re}\,(1-\omega)}\right).$$

If k is odd, then $\sigma_2(\gamma) = -2$. If k is even, let p be a prime dividing $k-1$; then $\sigma_p(\gamma) = -2$. The case $k > 0$ can be handled similarly. Also when $\ell < 0$ and $k = 0$, $\sigma_2(\gamma) = -2$.

Now assume $\ell > 0$; if γ is slice, then the Alexander polynomial of γ must satisfy the condition $\Delta(t) = \pm t^{\mu}\theta(t)\theta(t^{-1})$ (see [8]). Hence

$$\Delta(-1) = 4\ell - 3 + 4(-1)^k = \pm\theta(-1)^2.$$

Hence $\theta(-1)$ must be odd, say $\theta(-1) = 2b + 1$, so

$$4(\ell + (-1)^k - 1) + 1 = 4b(b+1) + 1,$$

and $\ell = b(b+1) + 1 + (-1)^{k+1}$. Therefore γ cannot be slice unless

$$\ell = b(b+1) + 1 + (-1)^{k+1}.$$

If $(\ell,k) \in \{(4,2n+1) : n \geq 1\}$ then

$$\Delta(t) = t^k(t^k - t^{k-1} - 3t + 7 - 3t^{-1} - t^{-(k-1)} + t^{-k}).$$

Plugging in $\omega_3 = e^{2\pi i/3}$, we get $\Delta(\omega_3) = \omega_3^k(t^k + t^{-k} - t^{k-1} - t^{-(k-1)} + 10)$. Then $|\Delta(\omega_3)| = 13$, 7, or 10 according to $k = 0,1,2 \pmod 3$. Hence (ℓ,k) cannot be slice in this range, for otherwise

$$|\Delta(\omega_3)| = |\theta(\omega_3)\,\theta(\omega_3^{-1})| = |\theta(\omega_3)\,\theta(\bar{\omega}_3)| = |\,\theta(\omega_3)\,\overline{\theta(\omega_3)}\,| = |\theta(\omega_3)|^2$$

(for some $\Theta(t)$).

Finally, the sliceness of γ is ruled out by section 4 when

$$(\ell,k) \notin \{(4,2n+1) : n \geq 0\}$$

and $\ell = b(b+1) + 1 + (-1)^{k+1} > 2$. So (ℓ,k) cannot be slice when

$$(\ell,k) \neq (0,0), (4,1), (2,k).$$

To prove the homotopy sliceness for the remaining cases we need the following.

LEMMA 1. *Let W^4 be a contractible manifold with boundary. Let $\gamma \subset \partial W^4$ be a knot such that $\pi_1(\partial W^4)/[\gamma] = \{1\}$ (where $[\gamma]$ is the smallest normal subgroup containing γ). Let α be a slice knot in $S^3 = \partial B^4$. If $\partial(\gamma^\circ) \approx \partial(\alpha^\circ)$, then γ is homotopically slice, where γ° is the 4-manifold obtained by attaching a 2-handle to W^4 along γ using the 0-framing.*

Proof. Let $D^2 \subset B^4$ be a 2-disc with $\alpha = \partial D^2 \subset S^3$.

Clearly the image of $i_\#: \pi_1(S^3 - \alpha) \to \pi_1(B^4 - D^2)$ normally generates $\pi_1(B^4 - D^2)$ where i is the inclusion. Define $V = B^4 - N(D^2)$ where $N(D^2)$ is an open tubular neighborhood of D^2 in B^4. Then $\partial V = \partial(\alpha^\circ)$ and since $i_\#[\alpha'] = 1$, where α' is the loop obtained by pushing off a copy of α with 0-framing, the image of the following map normally generates $\pi_1(V)$:

$$\pi_1(\partial(\alpha^\circ)) = \pi_1(\partial V) = \pi_1(S^3 - \alpha)/[\alpha'] \to \pi_1(V).$$

Define $\bar{W} = \partial W \times I \bigcup_{\gamma \times D^2} D^2 \times D^2 = \gamma^\circ - W, \quad \partial \bar{W} = \partial W \amalg \partial(\gamma^\circ).$

Let $\tilde{W} = \bar{W} \cup V$ where gluing is done using the diffeomorphism $\partial(\gamma^\circ) \approx \partial(\alpha^\circ)$. Then $\tilde{W}$ is a homology ball, and by Van Kampen's theorem

$$\begin{array}{ccc} & \pi_1(\partial(\alpha^\circ)) & \\ \swarrow & & \searrow \\ \pi_1(V) & & \pi_1(\bar{W}) \cong \pi_1(W)/[\gamma] \cong \{1\} \\ \searrow & & \nearrow \\ & \pi_1(\tilde{W}) & \end{array}$$

$\pi_1(\tilde{W}) = 0$; hence it is contractible. Also the knot $\gamma \subset \partial W = \partial \tilde{W}$ is slice because it bounds D^2 in $\bar{W}$, hence in $\tilde{W}$.

Let $(\ell, k) = (0,0), (4,1)$; then one can easily check that $\pi_1(\partial W^-(\ell, k))/[\gamma] = \{1\}$. For the other condition of Lemma 1, let $\gamma \subset \partial W^-(0,0)$; then from section 4, $\partial(\gamma^\circ) \approx \partial(O^\circ) = S^1 \times S^2$. If $\gamma \subset \partial W^-(4,1)$ then as in section 3 and section 4 we get $\partial(\gamma^\circ) \overset{\partial}{\approx} \partial(\alpha^\circ)$ where α is the stevedore's knot which is ribbon. Hence γ is homotopy slice when $(\ell, k) = (0,0), (4,1)$. It has been known for years that γ is homotopy slice when $(\ell, k) = (2, k)$. Here is a demonstration using the diffeomorphism

$$(\partial W^-(2,k), \gamma) \approx (\partial W^+(0, -k+1), \gamma)$$

and the moving pictures in Figure 17. The attaching circle of the 2-handle is concordant in $S^2 \times S^1 \times I$ to γ, so γ bounds a disk made up of the concordance and the core of the 2-handle.

Remark 6. The homotopy sliceness of $\gamma_- \subset \partial W^-(\ell, k)$ for

$$(\ell, k) = (0,0), (2,0), (2,1)$$

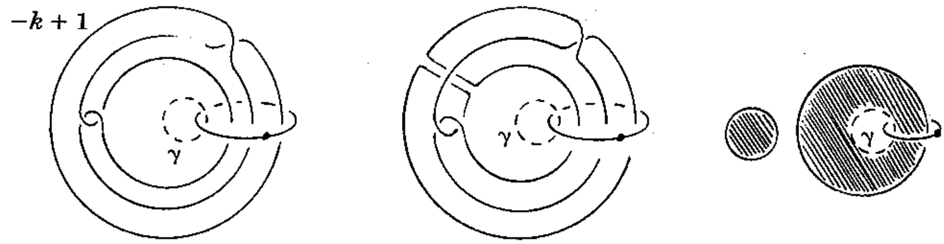

Figure 17

can be more geometrically seen as follows: Let $\gamma \subset \partial W^{-}(0,0)$. One can construct the diffeomorphism f in Figure 18 by first changing $S^1 \times B^3$ to $B^2 \times S^2$ in the interior of $W^{-}(0,0)$ (*i.e.,* removing the dot from the corresponding circle) and surgering the other imbedded S^2 (*i.e.,* putting a dot on the other circle). Clearly $f(\gamma)$ is slice in $W^{-}(0,0)$. Since $(W^{-}(2,1),\gamma) \approx (W^{+}(0,0),\gamma)$ the same trick works in this case.

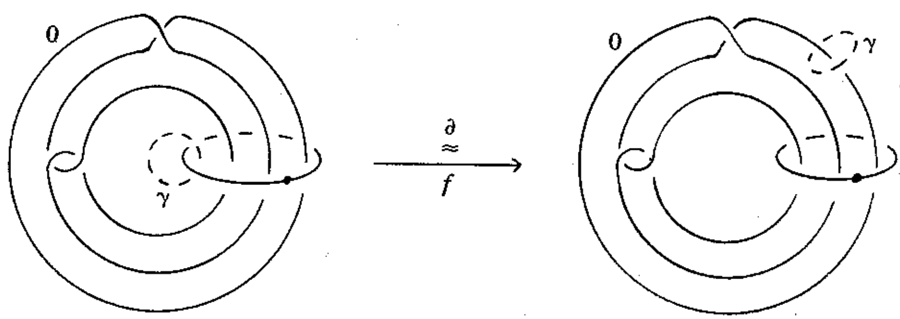

Figure 18

When $\gamma \subset \partial W^{-}(2,0) \approx \partial W^{+}(0,1)$, we construct in Figure 19 a diffeomorphism of $\partial W^{+}(0,1)$ taking γ to a slice knot.

Remark 7. Denote $\gamma \subset \partial W^{\pm}(\ell,k)$ by $\gamma_{\pm}(\ell,k)$. Then we have the following periodicity relation: $\sigma_p(\gamma_{\pm}(\ell,k+p)) = \sigma_p(\gamma_{\pm}(\ell,k))$.

Remark 8. One can generalize Mazur manifolds as follows: let $J \subset S^3$ be an oriented knot intersecting the interior of a 2-disc $D^2 \subset S^3$ algebraically once. We can construct a contractible manifold W_J^k by making ∂D^2 a 1-handle (*i.e.,* putting a dot on it) and attaching to J a 2-handle with framing k (Figure 20). Let $\gamma(J,k) \subset W_J^k$ be the first factor of $S^1 \times S^2$ (boundary of the 1-handle). Theorems 1, 1$'$ generalize to:

(i) $\gamma(J,0)$ is homologically slice if J is slice

(ii) $\sigma_p(\gamma(J,0)) = \sigma_p(J)$

(iii) $\sigma_p(\gamma(J, k+p)) = \sigma_p(\gamma(J,k))$

(ii), (iii) can be proved like Theorem 1, using a judicious choice of a Seifert

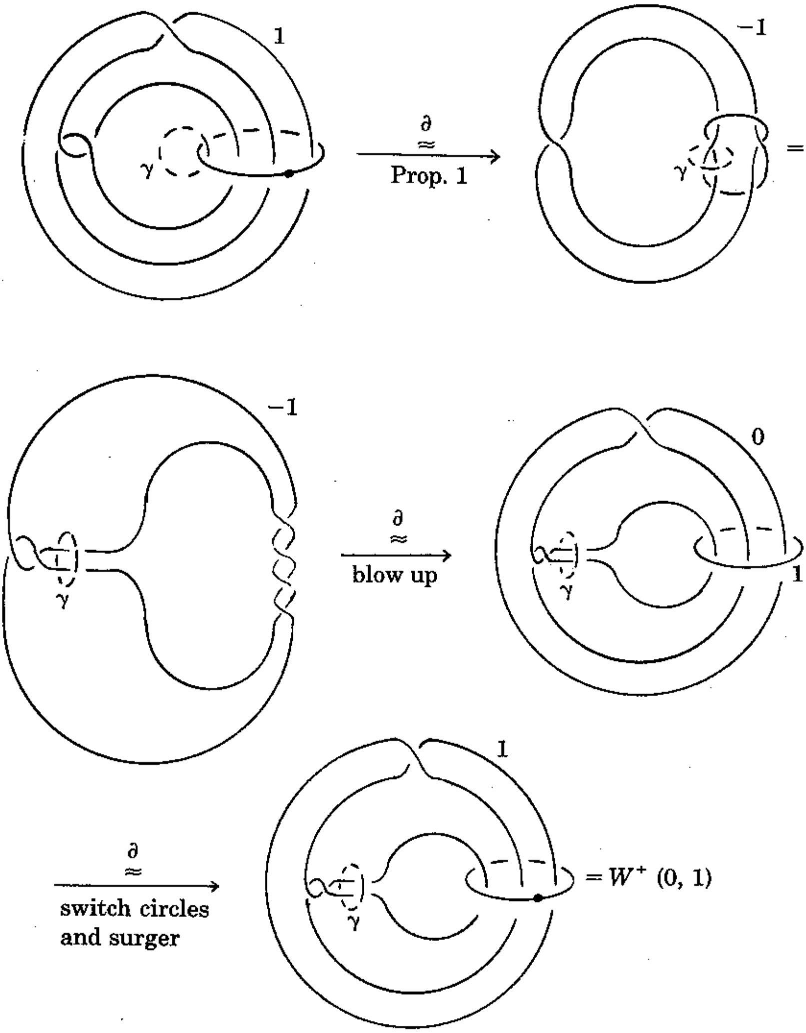

Figure 19

surface for $\gamma(J,k)$ as in [12]. (i) is true because $W_J^0 \cup D^2 \times D^2 = \gamma(J,0)^0 \overset{\partial}{\approx} J^0$ where $\gamma(J,0)^0$ is the 4-manifold obtained by attaching a 2-handle to W_J^0 along $\gamma(J,0)$ with 0-framing, then clearly $\gamma(J,0)$ is slice in the homology ball obtained by surgering the obvious $S^2 \hookrightarrow J^0$ in

$$(\partial W_J^0 \times I) \cup (D^2 \times D^2) \bigcup_{\partial} J^0.$$

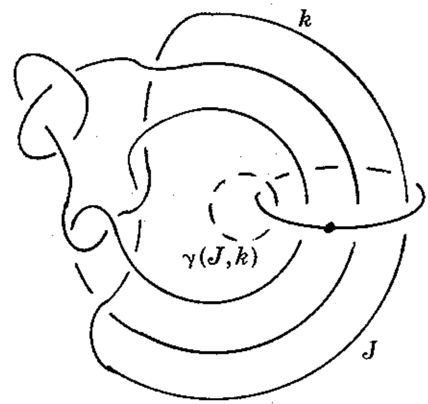

Figure 20

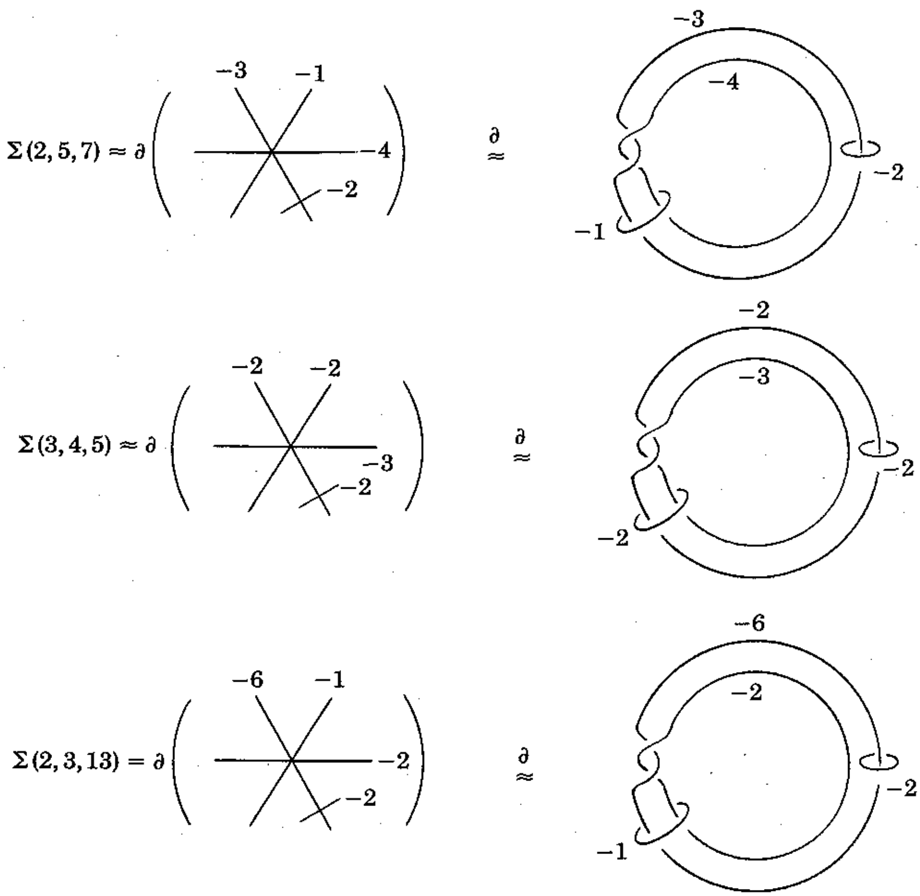

Figure 21

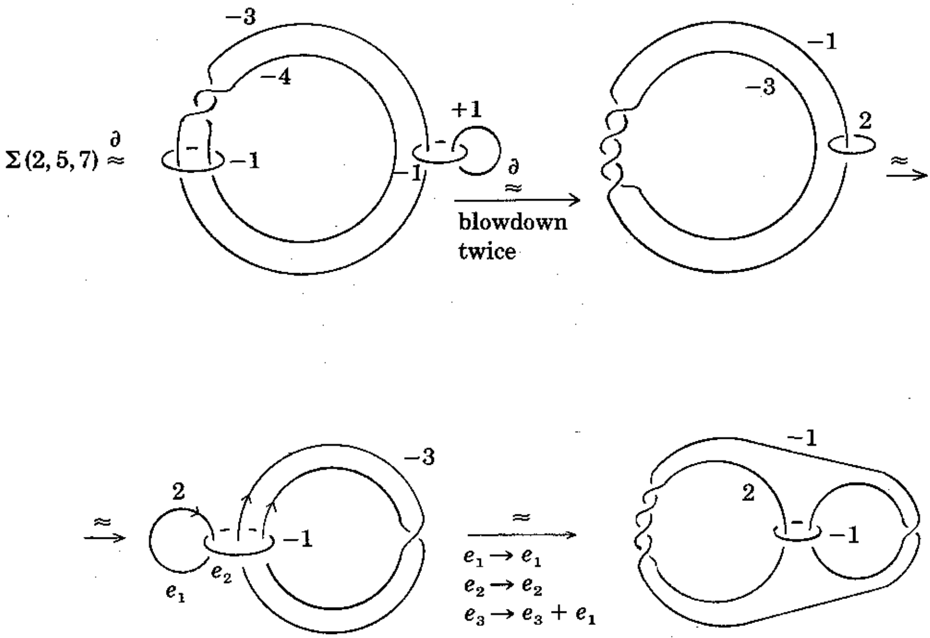

Figure 22

5. PROOF OF THEOREM 2

By resolving the singularity of these complex hypersurfaces as in [6] we get Figure 21. The intersecting line segments describe the plumbing description of 4-manifolds (see for example [6]). The above equalities can easily be verified. We are ready for the proofs:

In Figure 22, we start with $\Sigma(2,5,7)$, blow up once, blow down twice, do the indicated handle addition, and then blow down the small -1 circle and redraw to get $W^-(0,3)$. But $\partial W^-(0,3) \overset{\partial}{\approx} \partial W^-(1,2) \approx \partial W^-(2,1) \approx \partial W^+(0,0)$ by Proposition 1.

The case of $\Sigma(3,4,5)$ begins the same way, involves a similar handle slide, and is left to the reader.

In Figure 23, the first diagram is $\Sigma(2,3,13)$ after two -1 curves are blown down. After proceeding as indicated, take the last diagram, switch circles (the link is symmetric), surger the 0 circle, and reverse orientation. The result is $W^+(0,1)$ and $\partial W^+(0,1) \approx \partial W^+(1,0)$.

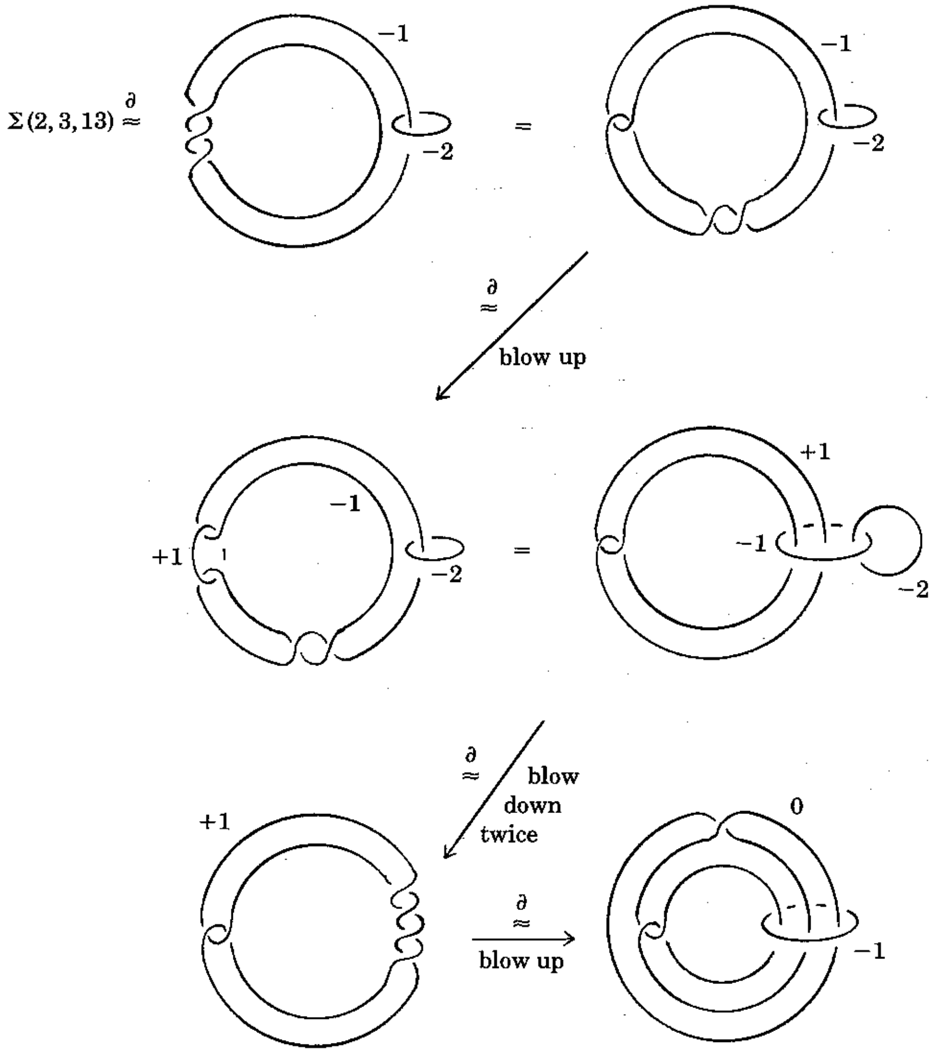

Figure 23

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