Math 1B
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1. Integral - by parts, trig, chain rule, partial fraction

(a) $\int \sin(\ln x)dx$

$$\begin{aligned}
\int \sin(\ln x)dx &= \int \sin u \cdot e^u du \qquad (u = \ln x, e^u = x, e^u du = dx) \\
&= e^u \sin u - \int e^u \cos u du \qquad \text{(integration by parts with } e^u \text{ and } \sin u) \\
&= e^u \sin u - \left[e^u \cos u - \int e^u(-\sin u)du \right] \quad \text{(int by parts again)} \\
&= e^u \sin u - e^u \cos u - \int e^u \sin u du
\end{aligned}$$

By moving $\int e^u \sin u du$ to the other side, we get

$$\int e^u \sin u du = \frac{1}{2}(e^u \sin u - e^u \cos u) + C$$

so we get

$$\int \sin(\ln x)dx = \frac{1}{2}(x \sin(\ln x) - x \cos(\ln x)) + C$$

(b) $\int \frac{e^{3x/2}}{e^x - 1}dx$

$$\begin{aligned}
\int \frac{e^{3x/2}}{e^x - 1}dx &= \int \frac{u^3}{u^2 - 1}\frac{2}{u}du \quad (u = e^{x/2}, du = \frac{e^{x/2}}{2}dx = \frac{u}{2}dx) \\
&= \int \frac{2u^2}{u^2 - 1}du \\
&= \int \left(2 + \frac{2}{u^2 - 1}\right) du \qquad \text{(by long division)} \\
&= u + \int \left(\frac{1}{u - 1} - \frac{1}{u + 1}\right) du \qquad \text{(by partial fraction)} \\
&= u + \ln|u - 1| - \ln|u + 1| + C \\
&= e^{x/2} + \ln|e^{x/2} - 1| - \ln|e^{x/2} + 1| + C
\end{aligned}$$

2. Proper/Improper integrals

Is $\int_0^\infty \frac{1}{x^{1/2}(x + 1)}dx$ convergent?

We have

$$\frac{1}{x^{1/2}(x+1)} \leq \min\{\frac{1}{x^{3/2}}, \frac{1}{x^{1/2}}\}$$

for $x > 0$, so

$$\int_0^\infty \frac{1}{x^{1/2}(x+1)}dx \leq \int_0^1 \frac{1}{x^{1/2}}dx + \int_1^\infty \frac{1}{x^{3/2}}dx < \infty$$

by p-series with $p < 1$ for $\int_0^1$ and $p > 1$ for $\int_1^\infty$.

3. Sequence/Series
(a) Does $a_n = n^{1/\sqrt{n}}$ converge? If yes, what is the limit?
By taking ln,

$$\lim_{n\to\infty} \ln n^{1/\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \ln n =^{L'hopital} \lim_{n\to\infty} \frac{1/n}{1/2\sqrt{n}} = 0$$

Hence, $\lim a_n = e^0 = 1$.

(b) Does $\sum_{n=1}^{\infty} \frac{n^{\sqrt{n}}}{2^n}$ converge?
By root test,

$$\left(\frac{n^{\sqrt{n}}}{2^n}\right)^{1/n} = \frac{n^{\sqrt{n}/n}}{2} = \frac{n^{1/\sqrt{n}}}{2} \to \frac{1}{2} < 1$$

by (a). Hence, it is convergent.
(c) Does $\sum_{n=0}^{\infty} \frac{n(-e)^n}{n^2e^n + 1}$ converge absolutely, conditionally?
We have alternating series,

$$\sum_{n=0}^{\infty} (-1)^n \frac{ne^n}{n^2e^n + 1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n + 1/(ne^n)}.$$

To test absolute convergence, by limit comparison test with $\sum_{n=0}^{\infty} \frac{1}{n}$,

$$\frac{\left(\frac{1}{n+1/(ne^n)}\right)}{\frac{1}{n}} = \frac{1}{1 + 1/(n^2e^n)} \to 1 > 0$$

so the given series does not converge absolutely. To test conditional convergence, we check that $\frac{1}{n+1/(ne^n)} \to 0$ and it is decreasing since its reciprocal $n + 1/(ne^n)$ is increasing:

$$n + 1/(ne^n) \leq n + 1 \leq (n+1) + 1/((n+1)e^{n+1}).$$

By alternating series test, it converges conditionally.

4. Interval of convergence of power series

Find interval of convergence of $\sum_{n=0}^{\infty}\frac{(x+1)^n}{2^n\sqrt{n}}$

$$\left|\frac{\frac{(x+1)^{n+1}}{2^{n+1}\sqrt{n+1}}}{\frac{(x+1)^n}{2^n\sqrt{n}}}\right| = \left|\frac{x+1}{2}\cdot\frac{\sqrt{n}}{\sqrt{n+1}}\right| \to \frac{|x+1|}{2} < 1$$

Hence, the radius of convergence is 2 and we have convergence for $|x+1| < 2$, or $-3 < x < 1$. To check the endpoints, at $x = -3$,

$$\sum_{n=0}^{\infty}\frac{(-2)^n}{2^n\sqrt{n}} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$$

converges by alternating series test. At $x = 1$,

$$\sum_{n=0}^{\infty}\frac{2^n}{2^n\sqrt{n}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$$

diverges by p-series. Hence, the interval of convergence is $[-3, 1)$

5. Separable differential equation
Solve $(2^x+1)y' = 2^{x+2y}$.
We have

$$(2^x+1)\frac{dy}{dx} = 2^x\cdot 2^{2y}$$
$$\frac{dy}{2^{2y}} = \frac{2^x dx}{2^x+1}$$
$$(1/4)^y dy = \frac{2^x dx}{2^x+1}$$

By integrating both sides, (note $(a^x)' = (\ln a)a^x$)

$$\frac{1}{\ln(1/4)}(1/4)^y = \frac{1}{\ln 2}\ln|2^x+1| + C$$

Since $\ln(1/4) = \ln 2^{-2} = -2\ln 2$,

$$(1/4)^y = -2\ln|2^x+1| + C'$$

Taking log,

$$y = \frac{1}{-2\ln 2}\ln(-2\ln|2^x+1| + C')$$

6. Linear differential equation (IVP/BVP)
(a) $y''+y = \cos x(1+2\sin x)$ $\quad y(0) = y(\pi/2) = 0$, Hint: double angle formula
Auxiliary equation: $r^2+1 = 0 \Rightarrow r = \pm i$. Hence $y_c = C_1\cos x + C_2\sin x$. Also RHS simplifies to

$$\cos x + 2\cos x\sin x = \cos x + \sin(2x)$$

so we need to find two particular solutions y_1 and y_2 corresponding to $\cos x$ and $\sin(2x)$, respectively. Our initial guess for y_1 is $A\cos x + B\sin x$ but since this is in complementary solution space, our modified guess is $x(A\cos x + B\sin x)$. We have:

$$y_1' = A\cos x + B\sin x + x(B\cos x - A\sin x)$$

$$y_1'' = 2B\cos x - 2A\sin x) + x(-A\cos x - B\sin x)$$

so by substituting to the equation,

$$y_1'' + y_1 = 2B\cos x - 2A\sin x = \cos x$$

i.e. $A = 0, B = 1/2$. For y_2, out guess is $C\cos(2x) + D\sin(2x)$, so

$$y_2' = 2D\cos(2x) - 2C\sin(2x)$$

$$y_2'' = -4C\cos(2x) - 4D\sin(2x)$$

$$y_2'' + y_2 = -3C\cos(2x) - 3D\sin(2x) = \sin(2x)$$

i.e. $C = 0, D = -1/3$. Putting this together, we get our general solution

$$y = C_1\cos x + C_2\sin x + \frac{1}{2}x\sin x - \frac{1}{3}\sin(2x)$$

Using $y(0) = y(\pi/2) = 0$, we get $C_1 = 0$, $C_2 + (1/2)(\pi/2) = 0$, so our final solution is

$$y = -(\pi/4)\sin x + \frac{1}{2}x\sin x - \frac{1}{3}\sin(2x)$$

(b) $y'' + 2y' + y = \dfrac{x}{e^{2x}} \qquad y(0) = 0, y'(0) = 1$

Auxiliary equation $r^2 + 2r + 1 = 0$ yields $r = -1$ (double root). Hence,

$$y_c = C_1e^{-x} + C_2xe^{-x}.$$

Since RHS is xe^{-2x}, our guess is $y_p = (Ax + B)e^{-2x}$.

$$y_p' = Ae^{-2x} - 2(Ax + B)e^{-2x}$$

$$y_p'' = -2Ae^{-2x} - 2Ae^{-2x} + 4(Ax + B)e^{-2x}$$

$$y_p'' + 2y_p' + y_p = Axe^{-2x} + (B - 2A)e^{-2x} = xe^{-2x}$$

i.e. $A = 1, B - 2A = 0$ so $B = 2$. Hence the general solution is

$$y = C_1e^{-x} + C_2xe^{-x} + (x + 2)e^{-2x}$$

with derivative

$$y' = -C_1e^{-x} + C_2e^{-x} - C_2xe^{-x} + e^{-2x} - 2(x + 2)e^{-2x}$$

Using initial condition, $C_1 + 2 = 0$, $-C_1 + C_2 + 1 - 4 = 1$, so $C_1 = -2, C_2 = 2$. Our final solution is

$$y = -2e^{-x} + 2xe^{-x} + (x+2)e^{-2x}.$$

7. Variation of parameters:
$y'' + y = \sec^3 x, 0 < x < \pi/2$ (Stewart 17.3, #24)
Auxiliary equation $r^2 + 1 = 0$ yields $y = \pm i$.

$$y_c = C_1 \cos x + C_2 \sin x.$$

We write

$$y_p = u_1 \cos x + u_2 \sin x$$

so that

$$y_p' = u_1' \cos x + u_2' \sin x - u_1 \sin x + u_2 \cos x$$

By letting

$$u_1' \cos x + u_2' \sin x = 0 \tag{1}$$

, we get

$$y_p'' = -u_1' \sin x - u_1 \cos x + u_2' \cos x - u_2 \sin x.$$

By substituting into equation,

$$y_p'' + y_p = -u_1' \sin x + u_2' \cos x = \sec^3 x = \frac{1}{\cos^3 x} \tag{2}$$

By equating $\sin x * (1) + \cos x * (2)$, we get

$$u_2' = \frac{1}{\cos^2 x}$$

$$u_2 = \int \sec^2 x dx = \tan x + C.$$

Also, since $u_1' = -u_2' \sin x / \cos x = -\sec^2 x \tan x$,

$$u_1 = -\int \sec^2 x \tan x dx = -\frac{\tan^2 x}{2} + C'$$

($-\frac{\sec^2 x}{2} + C''$ is also the solution. They are related by trig identity) Our final solution is (noting $\tan^2 x \cos x = \tan x \sin x$)

$$y = C_1 \cos x + C_2 \sin x - \frac{\tan^2 x}{2} \cos x + \tan x \sin x = C_1 \cos x + C_2 \sin x + \frac{\tan x \sin x}{2}.$$

8. Word problems - spring/work, hydro force, mixing, damping

9. Centroid/arc length

What is the centroid/perimeter of the region enclosed by $y = x^2$ and $y = 2x + 3$?
First, we find that the two lines intersect at $x^2 = 2x + 3 \Rightarrow (x - 3)(x + 1) = 0$, i.e. $(x, y) = (-1, 1)$ and $(3, 9)$. The area is

$$A = \int_{-1}^{3} 2x + 3 - x^2 dx = [x^2 + 3x - x^3/3]_{-1}^{3} = 32/3$$

Hence,

$$\begin{aligned}
\bar{x} &= A^{-1} \int_{-1}^{3} x(2x + 3 - x^2)dx \\
&= A^{-1}[2x^3/3 + 3x^2/2 - x^4/4]_{-1}^{3} \\
&= (45/4 - 7/12)A^{-1} = 3/2 \\
\bar{y} &= A^{-1} \int_{-1}^{3} \frac{1}{2}((2x + 3)^2 - x^4 dx)dx \\
&= (2A)^{-1}[4x^3/3 + 6x^2 + 9x - x^5/5]_{-1}^{3} \\
&= (2A)^{-1}(342/5 - (-62/15)) = 17/5
\end{aligned}$$

For arc length,

$$\int_{-1}^{3} \sqrt{1 + (2x)^2}dx = \frac{1}{2}\int_{-1}^{3} \sqrt{1/4 + x^2}dx = \frac{1}{2}\left[\frac{x}{2}\sqrt{1/4 + x^2} + \frac{1}{8}\ln\left(x + \sqrt{1/4 + x^2}\right)\right]_{-1}^{3}$$

using Formula 21 Appendix p6, Stewart, or using trig substitution $x = \frac{1}{2}\tan\theta$. The straight line portion of perimeter is $4\sqrt{5}$ so the perimeter is

$$\left(\frac{3}{8}\sqrt{37} + \frac{1}{8}\sqrt{5} + \frac{1}{16}\ln(3 + \frac{\sqrt{37}}{2}) - \frac{1}{16}\ln(\frac{\sqrt{5}}{2} - 1)\right) + 4\sqrt{5}$$

10. Taylor series
(a) Find Maclauren series for $f(x) = \frac{\ln(1+x)}{x}$. What is $f^{(3)}(0)$?

$$\ln(1 + x) = \int \frac{1}{1 + x}dx = \int \sum_{n=0}^{\infty}(-1)^n x^n dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n + 1}$$

Note the constant of integration is 1 by evaluating both sides at 1. Since $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$, $\frac{f^{(3)}(0)}{3!}$ is the coefficient for x^3 in the series which is 1/3 from above. Hence, $f^{(3)}(0) = 3!/3 = 2$.

(b) Find Taylor polynomial for $x^{2/3}$ at $a = 8$ so that the error is less than 10^{-2} for $7 < x < 9$.

$$\begin{aligned}
f'(x) &= \left(\frac{2}{3}\right) x^{-1/3} \\
f''(x) &= \left(\frac{2}{3}\right)\left(\frac{-1}{3}\right) x^{-4/3} \\
f'''(x) &= \left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right) x^{-7/3}
\end{aligned}$$

Hence, we have

$$\begin{aligned} f(x) &= f(8) + f'(8)(x-8) + \frac{f''(8)}{2}(x-8)^2 + R_2(x) \\ &= 4 + \frac{1}{3}(x-8) - \frac{1}{72}(x-8)^2 + R_2(x) \end{aligned}$$

But we have

$$|R_2(x)| = \left|\frac{f'''(z)}{3!}(x-8)^3\right| = \left|\frac{8}{27z^{-7/3}\cdot 6}(x-8)^3\right| < \left|\frac{4}{81z^{7/3}}\right|$$

where the last inequality holds for $7 < x < 9$. Also, $\left|\frac{4}{81z^{7/3}}\right| < \frac{4}{81\cdot 7^{7/3}} < 10^{-2}$ so $f(x) \approx 4 + \frac{1}{3}(x-8) - \frac{1}{72}(x-8)^2$ to the desired accuracy.
(c) Estimate the error of approximation $\ln(0.8) \approx -0.2$ using Taylor's formula.
(cf. Serganova, Sample final #2, Prob 10).

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} = x - x^2/2 + \dots$$

$$\ln(1-0.2) = -0.2 + R_1(0.2)$$

where $|R_1(0.2)| = \left|\frac{f''(z)}{2}(-0.2)^2\right|$ for some $-0.2 < z < 0$. Since $f''(x) = -\frac{1}{(1+x)^2}$, we have

$$|R_1(0.2)| = \left|\frac{0.2^2}{2(1+z)^2}\right| < \frac{0.2^2}{2(1-0.2)^2} = 1/32$$

for $-0.2 < z < 0$. Hence, error is less than 1/32.

(d) Estimate $\sin 0.1$ within 10^{-5} using alternating series test.

$$\sin x = x - x^3/3 + x^5/5 + \dots$$

so $\sin 0.1 \approx 0.1 - 0.1^3/3$ with error less than $0.1^5/5$ by alternating series test. Hence the approximation $\sin 0.1 \approx 0.1 - 0.1^3/3$ meets the specified accuracy.