

L1 - Introduction to

A) Regularization

B) Moduli spaces of pseudoholomorphic curves

survey-references:

\* Polyfolds - A first and second glance, Fuchs - Fish -  
Golovko - W.  
parts of §2, §3

\* Smooth Kuranishi atlases, McDuff - W  
parts of §2, §3

Regularization Thm 0

$E$   $B$  finite dim manifold  
 $\downarrow$   $\left. \begin{array}{l} \\ \end{array} \right\} s$   $E$  finite rank vector bundle  
 $B$   $s \in C^\infty$  section,  $s^{-1}(0)$  compact

$\Rightarrow \exists \mathcal{P} \subset C_0^\infty(B, E) := \{p : B \rightarrow E \text{ section w. compact support}\}$   
 s.t.

(nontrivial) :  $\exists (p_i)_{i \in \mathbb{N}} \subset \mathcal{P} : p_i \rightarrow 0 \in C_0^\infty$

(transversality)  $\forall p \in \mathcal{P}$   $s+p \neq 0$  i.e.  $\forall b \in (s+p)^{-1}(0) : D_b(s+p) : T_b B \rightarrow E_b$  surjective  
 $(\Rightarrow (s+p)^{-1}(0) \subset B$  submanifold)

(compactness)  $\forall p \in \mathcal{P}$   $(s+p)^{-1}(0)$  compact

(cobordism)  $\forall p, p' \in \mathcal{P} \exists$  cobordism  $(s+p)^{-1}(0) \sim (s+p')^{-1}(0)$

$\hookrightarrow$  compact manifold with boundary  
 $(s+p)^{-1}(0) \sqcup (s+p')^{-1}(0)$

In fact,

$\mathcal{P} = \{p \in C_0^\infty(B, E) \mid s+p \neq 0\}$

(not discussing orientations here)

Cor.:  $[(s+p)^{-1}(0)] \in \text{Bord}_d$  ("d-manifolds modulo cobordism")  
 $[(s+p)^{-1}(0)] \in H_d(B)$  (almost any homology notion)  
 are well defined (i.e. independent of  $p \in \mathcal{P}$ )

Rmk: We can then use inverse limit properties of rational Čech homology to construct a (not so 'virtual') fundamental class  $[\bar{M}] \in H_d(\bar{M})$  of the possibly singular but compact metric space  $\bar{M} := s^{-1}(0)$ .

(McDuff-W)  $\stackrel{= s^{-1}(0)}{=} \stackrel{= B}{=}$   
 Remark 7.5.3. Let  $X$  be a compact subset of a metric space  $Y$ , and let  $(W_k \subset Y)_{k \in \mathbb{N}}$  be a sequence of open subsets that is nested,  $W_k \subset W_{k-1}$ , such that  $X = \bigcap_{k \in \mathbb{N}} W_k$ . Then the system of maps  $\check{H}_n(X; \mathbb{Q}) \rightarrow \check{H}_n(W_{k+1}; \mathbb{Q}) \rightarrow \check{H}_n(W_k; \mathbb{Q})$  induces an isomorphism

$$\check{H}_n(X; \mathbb{Q}) \xrightarrow{\cong} \varprojlim \check{H}_n(W_k; \mathbb{Q}).$$

$$\mathcal{W}_k = \{b \in B \mid d(b, s^{-1}(0)) < 1/k\}$$

pick  $p_k \in \mathcal{P}$  s.t.  $(s+p_k)^{-1}(0) \subset \mathcal{W}_k$

$$[(s+p_k)^{-1}(0)] = [(s+p_{k+1})^{-1}(0)]$$

$\Downarrow$

$$[s^{-1}(0)] := \varprojlim [(s+p_k)^{-1}(0)] \quad \text{well defined}$$

Regularization Thm 0

$$\begin{array}{c} E \\ \downarrow \\ B \end{array} \Bigg\} s$$

$C^\infty$  section of finite dim bundle  
 $s^{-1}(0)$  compact

$\Rightarrow \exists \mathcal{P} \subset C_0^\infty(B, E)$  (nontrivial) (transversality) (compactness) (cobordism)

generalizes to \* Fredholm sections [using Sard-Smale]

$$G \Bigg\} \begin{array}{c} E \\ \downarrow \\ B \end{array} \Bigg\} s$$

\*  $G$ -equivariance for  $G$  finite

$\rightarrow p$  "multisections"

$\rightarrow \frac{(s+p)^{-1}(0)}{G}$  "weighted branched mfd"

$$\Rightarrow [\sim] \in H_d(B/G; \mathbb{Q})$$

does not generalize to

•  $G$  nondiscrete ( $s+p \neq 0 \Leftarrow pg = gp$ )

•  $s^{-1}(0)$  noncompact (transv.  $\checkmark$  cobordism  $\checkmark$ )

Ex 1

$$\mathbb{R} \times \mathbb{R}$$

$$\downarrow \uparrow s(x) = (x, 0)$$

$$\mathbb{R}$$

$$s^{-1}(0) = \mathbb{R} \text{ noncompact}$$

$$p(x) = x \Rightarrow (s+p)^{-1}(0) = \{0\}$$

$$p'(x) = \varepsilon \neq 0 \Rightarrow (s+p')^{-1}(0) = \emptyset$$

not cobord.

Ex 2

$$S^1 \hookrightarrow \mathbb{C}$$

$$\psi$$

$$e^{i\theta}$$

$$\mathbb{C} \times \mathbb{R}$$

$$\downarrow \uparrow s(z) = (z, |z|^2)$$

$$\mathbb{C}$$

$$s^{-1}(0) = \{ |z|=0 \} = \{0\} = pt$$

$$\psi$$

$$z \mapsto e^{i\theta} z$$

$$p = \varepsilon > 0$$

$$p = -\varepsilon < 0$$

$$s \neq 0$$

$$(s+p)^{-1}(0) = \{ |z|^2 = -\varepsilon \} = \emptyset$$

$$(s+p')^{-1}(0) = \{ |z|^2 = \varepsilon \} = pt$$

