UNCERTAINTY PRINCIPLE, NON-SQUEEZING THEOREM AND THE SYMPLECTIC RIGIDITY

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Yong-Geun Oh, Department of Mathematics
University of Wisconsin Madison, WI 53706

§1. Prologue: Uncertainty principle and non-squeezing theorem.

One of the basic principle in the quantum mechanics is the Heisenberg uncertainty principle. This can be roughly stated as "one can not measure the momentum and the position of a particle precisely at the same time". More precisely, the principle can be written as

$$\triangle q \triangle p \simeq \hbar, \quad \hbar = \text{the Planck constant},$$

where $\triangle q = \langle q - \langle q \rangle \rangle, \triangle p = \langle p - \langle p \rangle \rangle$ are the deviation from the average values $\langle q \rangle$ and $\langle p \rangle$. When a particle is in $\mathbb{R}^3$, then this is replaced by

$$\triangle q_i \triangle p_i \simeq \hbar \tag{1.1}$$

for $i = 1, 2, 3$. A natural question then is to ask what would be the analogue in the classical mechanics. This question involves two tasks in it: First we need to *formulate* what would be the statement and secondly need to *prove* the statement.

To formulate the analogue, we need some digression into the *Hamiltonian formalism* of the classical mechanics.

1.1. Hamiltonian mechanics.

Newtonian mechanics is based on the *Newton's second law of motion*:

$$m\ddot{x} = F(x) \quad x \in \mathbb{R}^3 \tag{1.2}$$

This is a second order ordinary differential equation of x. To determine the motion of a particle, we need two initial conditions

$$\begin{cases} x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases} \tag{1.3}$$

One can transform (1.2) and (1.3) into a system of first order ODE on $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$

$$\begin{cases} \dot{q} = p \\ \dot{p} = F(q) \\ q(0) = x_0 \quad p(0) = v_0 \end{cases} \tag{1.4}$$

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where we substitute $q = x \quad p = m\dot{x}$. From now on, we will set $m = 1$.

One can easily check that the total energy

$$E = E(q,p) := \frac{1}{2}m\dot{x}^2 - \int F(x)$$
$$= \frac{1}{2}p^2 - \int F(q)$$

is conserved along each trajectory of (1.4).

With the function $H = E(q,p)$, the equation (1.4) can be written as

$$\text{(1.5)} \qquad \begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

which is called the *Hamilton's equation* associated to the Hamiltonian H. One can think of the equation (1.5) for any given smooth function H. The same conservation law then holds for (1.5), i.e., the *Hamiltonian* H is conserved along the trajectory of (1.5).

There could be other type of *conserved quantities* depending on the type of the Hamiltonian H. For example, when for the Hamiltonian $H = \frac{1}{2}p^2 - \int F(q)$, the force F is rotationally symmetric, then the *angular momentum* will be conserved. In general, each symmetry of the mechanical system gives rise to a conserved quantity –"Nöether's principle"–.

In the hamiltonian mechanics, the conserved quantities play an important role in solving the given mechanical system and it is important to find out as many conserved quantities as possible. For this purpose, the so called *Poisson bracket* is a very useful tool, which turns out to be the essential geometric structure when one study the mechanics with constraint, i.e., *mechanics on manifolds*.

Let us first recall the definition of classical Poisson bracket on $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$.

Definition 1.1 (The classical Poisson bracket). For each pair (G,H) of smooth function on $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$, we define a new function, denoted by $\{G,H\}$, by

$$\text{(1.6)} \qquad \{G,H\} = \sum_{j=1}^{3}\left(\frac{\partial G}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial G}{\partial p_j}\frac{\partial H}{\partial q_j}\right)$$

The followings are the main properties of this bracket, which can be proven by direct calculation:

(1) $\{G,H\} = -\{H,G\}$ –"Skew-symmetry"
(2) $\{G,H_1 + H_2\} = \{G,H_1\} + \{G,H_2\}$ –"Bilinearity"
(3) $\{G,HF\} = \{G,H\}F + H\{G,F\}$ –"Leibniz rule"
(4) $\{G,\{H,F\}\} + \{H,\{F,G\}\} + \{F,\{G,H\}\} = 0$ –"Jacobi identity"

The following theorem illustrates the usefulness of the Poisson bracket.

Theorem 1.2. *If F and G are conserved quantities of the Hamilton's equation (1.5), then $\{F, G\}$ also becomes a conserved quantity.*

Proof. This immediately follows from the identity

$$\frac{d}{dt} G \circ \gamma(t) = \{H, G\}(\gamma(t)) \tag{1.7}$$

where γ is a solution of the Hamilton's equation

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

The identity (1.7) can be proven by a direct calculation from the definition (1.6). □

In the physics literature (e.g. [Go]), a mapping (or coordinate change) $\phi : \mathbb{R}^6 \to \mathbb{R}^6$ is called a *canonical transformation* if the identity

$$\{G \circ \phi, H \circ \phi\} = \{G, H\} \circ \phi \tag{1.8}$$

holds for any smooth functions G and H. In physics literature, (1.8) is usually written as

$$\sum_{j=1}^{3} \left(\frac{\partial G}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial G}{\partial p_j}\frac{\partial H}{\partial q_j}\right) = \sum_{j=1}^{3} \left(\frac{\partial \overline{G}}{\partial Q_j}\frac{\partial \overline{H}}{\partial P_j} - \frac{\partial \overline{G}}{\partial P_j}\frac{\partial \overline{H}}{\partial Q_j}\right)$$

(sometimes even without "bar"), where one denote

$$\phi(q, p) = (Q(q, p), P(q, p))$$
$$\overline{G}(Q, P) = G(q(Q, P), p(Q, P))$$

Lioville's theorem. *Any canonical transformation preserves the phase volume in $\mathbb{R}^6$.*

We will give the proof of this theorem later, using other equivalent definition of the canonical transformation.

Example 1.3. Consider a compactly supported time dependent Hamiltonian

$$H : [0, 1] \times \mathbb{R}^6 \to \mathbb{R}$$

and consider the non-autonomous Hamiltonian equation

$$\begin{cases} \dot{q} = \dfrac{\partial H_t}{\partial p} \\ \dot{p} = -\dfrac{\partial H_t}{\partial q} \quad H_t(q, p) := H(t, q, p) \end{cases} \tag{1.9}$$

and its *time-one map* $\phi_1^H : \mathbb{R}^6 \to \mathbb{R}^6$. More precisely, consider the solution $\gamma : \mathbb{R} \to \mathbb{R}^6$ of (1.9) with the initial condition $\gamma(0) = (q_0, p_0)$. We denote this solution by $\gamma_{(q_0, p_0)}$. And then define

$$\phi_1^H(q_0, p_0) = \gamma_{(q_0, p_0)}(1).$$

By the standard theorem of ODE, ϕ_1^H defines a smooth diffeomorphism.

Proposition 1.4. *The* $\phi_1^H : \mathbb{R}^6 \to \mathbb{R}^6$ *is a canonical transformation.*

One immediate consequence of this proposition is that the set of canonical transformations is *infinite dimensional*.

1.2. The classical analogue: Non squeezing theorem

One can easily check that the discussions in the previous section is generalized to the even dimensional space $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. One can also consider $\mathbb{R}^{2n}$ as the n-product of $\mathbb{R}^2 \simeq \mathbb{C}$, i.e.,

$$\mathbb{R}^{2n} = \mathbb{R}^2 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \simeq \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C} = \mathbb{C}^n.$$

In this identification, we use the correspondence

$$(q_j, p_j) \longleftrightarrow q_j + ip_j =: z_j \quad j = 1, \cdots, n.$$

Now, one can formulate the classical analogue of the uncertainty principle (1.1) as follows, which is due to Arnold and Gromov: Consider the standard R-ball in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$

$$B^{2n}(R) = \{z \in \mathbb{C}^n \mid \sum_{j=1}^n |z_j|^2 \le R^2\}$$

and the cylinder over z_1-plane

$$Z^{2n}(r) = \{z \in \mathbb{C}^n \mid |z_1|^2 \le r^2\}.$$

Note that $B^{2n}(R)$ has *finite volume* while $Z^{2n}(r)$ has *infinite volume* and so one can easily see that one can embed $B^{2n}(R)$ into $Z^{2n}(r)$ by a *volume preserving* map, whatever r and R are. One might ask the question whether such an embedding is possible by *canonical transformations* which are a subclass of volume preserving maps.

Non-squeezing theorem [Gromov, Gr]. *If* $r < R$*, we cannot embed* $B^{2n}(R)$ *into* $Z^{2n}(r)$ *by any canonical transformation.*

Physically speaking, this theorem says that if a collection of particles initially spread out all over the unit ball $B^{2n}(R)$, then one cannot squeeze the collection into a statistical state in which the momentum and position in the (q_1, p_1)-direction spreads out less than initially.

The proof [Gr] of this simply stated principle turned out to require a completely new technique in the symplectic geometry at the time when Gromov's paper [Gr] appeared. It requires the combination of several different disciplines of mathematics, e.g., differential geometry, complex analysis, index theory, partial differential equation and etc.. This is the prototype of the current trend of the 3-dimensional and 4-dimensional differential topology as well. The above non-squeezing theorem is the major starting point of the area, the *symplectic topology*, more specifically the *symplectic rigidity theory*.

§2. Symplectic Manifolds

Recall that the classical Poisson bracket $\{\ ,\ \}$ on $C^\infty(\mathbb{R}^{2n})$ defines a *Poisson algebra* on $C^\infty(\mathbb{R}^{2n})$, i.e., defines a bilinear map

$$\{,\} : C^\infty(\mathbb{R}^{2n}) \times C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n})$$

which satisfies

(1) $\{G, H\} = -\{H, G\}$
(2) $\{G, H_1 + H_2\} = \{G, H_1\} + \{G, H_2\}$
(3) $\{G, HF\} = \{G, H\}F + H\{G, F\}$
(4) $\{G, \{H, F\}\} + \{H, \{F, G\}\} + \{F, \{G, H\}\} = 0$

Now, one can take these 4 properties as axioms for the so called *Poisson manifold*.

Definition 2.1. A *Poisson manifold* is a pair $(P, \{,\})$ where $\{\ ,\ \}$ defines a bilinear map on $C^\infty(P)$ satisfying (1) - (4). We call $\{\ ,\ \}$ a *Poisson bracket*.

Examples 2.2.

(1) Any manifold with trivial bracket, i.e., $\{H, F\} = 0$ for all F, H.– "ordinary manifold"
(2) The classical phase space $\mathbb{R}^{2n}$ with the classical bracket.

Now, let us derive some consequences from the above 4 axioms. First, we have

Lemma 2.3. *For any $H \in C^\infty(P)$ and a constant function C, we have*

$$\{H, C\} = 0$$

Note that (2) and (3) imply that for any given H, the linear map

$$\{H, \cdot\} : C^\infty(P) \to C^\infty(P)$$

becomes a derivation and so is associated to a vector field, denoted by X_H, which is called the *Hamiltonian vector field* associated to H.

Furthermore Lemma 2.3 implies that the value $X_H(p) \in T_pP$ depends only on $dH(p) \in T_p^*P$ but does not depend on the choice of H (**Exercise 1**). Therefore the assignment

$$dH(p) \mapsto X_H(p)$$

defines a bundle map

$$\pi : T^*P \to TP.$$

We refer the readers to [W] for basic properties of general Poisson manifolds. In the example 2.2 (1), this map is the zero map and in 2.2 (2), it becomes a bundle isomorphism. In this lecture, we will be interested only in the case when the above bundle map π is an isomorphism. When this map is an isomorphism, we denote its inverse by

$$\widetilde{\omega} : TP \to T^*P$$

in an obvious reason, which will be clear from below. In this case, we can interpret the assignment

$$p \mapsto \widetilde{\omega}_p$$

as a covariant two-tensor, denoted by ω, that is *skew-symmetric* which follows from the skew-symmetry (1) of the Poisson bracket:

$$\omega_p(v, w) := \widetilde{\omega}_p(v)(w)$$

This means that ω is a *differential two-form*. Furthermore ω is *nondegenerate* because we assume that $\widetilde{\omega}_p$ is an isomorphism at each $p \in P$. Finally, one can prove

Exercise 2. Prove that when $\pi : T^*P \to TP$ is a bundle isomorphism, the Jacobi identity (4) of $\{\ ,\ \}$ is equivalent to the *closedness* of the two form ω.

Definition 2.4. A *symplectic manifold* is a pair (P, ω) where P is a manifold and ω is a nondegenerate closed two form. We call ω the *symplectic form*, or the *symplectic structure.*

Examples 2.5.

(1) The classical phase space $(\mathbb{R}^{2n}, \omega_0)$ associated to the classical Poisson bracket. In this case, one can show (**Exercise 3**)

$$\omega_0 = \sum_{j=1}^{n} dq_j \wedge dp_j.$$

(2) More generally, for any smooth manifold M, its cotangent bundle T^*M carries a canonical symplectic structure

$$\omega_0 = -d\theta$$

where θ is the canonical one-form defined by

$$\theta_p(\xi) = p(T\pi\xi)$$

for each $p \in T^*M$ and $\xi \in T_p(T^*M)$, where $\pi : T^*M \to M$ is the canonical projection. In the *canonical coordinates* of T^*M, θ is expressed as

$$\theta = \sum_{j=1}^{n} p_j dq_j.$$

Note that the above case (1) is a special case of this by considering $\mathbb{R}^{2n}$ as $T^*\mathbb{R}^n$. The above cases are examples of non-compact ones.

(3) Any two dimensional (compact) oriented surfaces with an area form.

(4) Any Kähler manifold with the Kähler form, e.g., $\mathbb{C}P^n$ or any algebraic manifolds.

(5) [Gompf, Gm] *Any finitely presented group π (i.e., finitely generated with finite relations) can be realized as the fundamental group of compact symplectic four-manifolds.*

One of the main current themes in the four dimensional differential topology is to understand the role of compact symplectic 4-manifolds in the classification problem of (simply connected) 4-dimensional differentiable manifolds. Some topologists speculate that symplectic manifolds are one of the building blocks of 4-manifolds. (See [Gm], [T1,2,3] in relation to this aspect.)

The following theorem shows that unlikely from the case of Riemannian manifolds where the curvatures provides the *local* invariants of the metric, symplectic manifolds have no local invariants.

Definition 2.6. A map $\phi : (P_1, \omega_1) \to (P_2, \omega_2)$ is called a *symplectic map* if $\phi^*\omega_2 = \omega_1$ and a symplectic diffeomorphism (or symplectomorphism) when it is a diffeomorphism in addition.

Darboux Theorem. *Let $p \in (P, \omega)$ be any point. Then there exist a local chart (ϕ, U) at p, $\phi : U \to V \subset \mathbb{R}^{2n}$ such that*

$$\phi^* \omega_0 = \omega, \quad \text{i.e.},$$
$$\omega = \sum_{j=1}^{n} dq_j \wedge dp_j$$

in the coordinates $\phi = (q_1, \cdots, q_n, p_1, \cdots, p_n)$. In other words, any symplectic manifold (P^{2n}, ω) is locally symplectomorphic to the classical phase space $(\mathbb{R}^{2n}, \omega_0)$.

Exercise 4. Prove that in the classical phase space a map $\phi : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$ is symplectic iff ϕ is a canonical transformation, i.e., preserves the canonical Poisson bracket.

Remark 2.7. A symplectic manifold (P, ω) carries a canonical orientation provided by the form

$$\frac{1}{n!}\omega^n$$

that is a nondegenerate $2n$-form, i.e., a volume form. This volume form is called the *Lioville volume form* and nothing but the ordinary volume form in the classical phase space $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. By definition, any symplectic map $\phi : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$ (and so any canonical transformation) preserves the phase volume (Lioville's theorem).

Now, we can restate the non-squeezing theorem

Non-squeezing Theorem. *If $R > r$, there is no symplectic embedding*

$$\phi : (B^{2n}(R), \omega_0) \to (Z^{2n}(r), \omega_0)$$

§3. Pseudo-holomorphic Curves

We further analyze the statement of the non-squeezing theorem. It uses more than a symplectic structure, i.e., uses other geometric structures of $\mathbb{R}^{2n}$, *Euclidean metric* and the splitting of $\mathbb{R}^{2n} \simeq \mathbb{C}^n = \mathbb{C} \oplus \mathbb{C}^{n-1}$, which uses the *complex structure*. Note that $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ has three canonical bilinear forms: the symplectic form ω, the *Euclidean inner product* $g(,)$ and the Hermitian inner product $\langle \cdot, \cdot \rangle$. The relation between these three can be written as

$$\langle \cdot, \cdot \rangle = g(\cdot, \cdot) + i\omega(\cdot, \cdot).$$

From this, one can easily check that

$$g(\cdot, \cdot) = \omega(\cdot, J_0 \cdot) \tag{3.1}$$

where $J_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the standard (integrable) almost complex structure associated to the multiplication by i on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, i.e., J_0 is the endomorphism determined by

$$J_0(\frac{\partial}{\partial q_j}) = \frac{\partial}{\partial p_j}, \quad J_0(\frac{\partial}{\partial p_j}) = -\frac{\partial}{\partial q_j}$$

for $j = 1, \cdots, n$.

Now, we generalize (3.1) to arbitrary symplectic manifolds (P, ω).

Definition 3.1. An almost complex structure J on (P, ω) is called *compatible* (or *calibrated*) to ω if the bilinear form $\omega(\cdot, J\cdot)$ defines a positive definite symmetric bilinear form, i.e., Riemannian metric. We denote by $\mathcal{J}_\omega$ the set of such almost complex structures.

Proposition 3.2. [Gr]. *The set $\mathcal{J}_\omega$ is a contractible infinite dimensional (Frêchet) manifold.*

An ingenious idea of Gromov is to study the space of almost complex maps

$$f : (\Sigma, j) \to (P, J)$$

where (Σ, j) is another almost complex manifold. However, unless both j and J are integrable, there is no even local existence theorem of such map possible when $\dim \Sigma > 2$. On the other hand, when $\dim \Sigma = 2$, there is the following local existence theorem.

Definition 3.3. Assume $\dim \Sigma = 2$. An almost complex map $f : (\Sigma, j) \to (P, J)$ is called a (j, J)-holomorphic map.

In this paper, the main interest of ours will be the cases where $\Sigma = S^2$ or D^2 in which case we will suppress the complex structure j and call f *J-holomorphic map*, and its image a *J-holomorphic curve*.

Nijenhuis-Wolf [NW]. *At each point $p \in P$ and for each complex tangent 2-plane τ_p at p, there exists $\epsilon > 0$ and a J-holomorphic disc*

$$f : (D^2(\epsilon), J_0) \to (P, J)$$

such that $f(0) = p$ and $\tau_p = \text{Image } T_0 f$.

The proof of this theorem relies on the elliptic theory of PDE noting that the almost complex condition

$$Tf \circ j = J \circ Tf \tag{3.2}$$

is a quasi-linear first order elliptic partial differential equation. In fact, for the case $(D^2(\epsilon), J_0)$, (3.2) is equivalent to

$$\frac{\partial f}{\partial x} + J(f)\frac{\partial f}{\partial y} = 0 \tag{3.3}$$

where $\frac{\partial f}{\partial x} := Tf(\frac{\partial}{\partial x})$, $\frac{\partial f}{\partial y} := Tf(\frac{\partial}{\partial y})$ and (x, y) is the standard coordinate of $D^2(\epsilon)$. Of course, when J is integrable, (3.3) is exactly the classical Cauchy-Riemann equation.

Now, we study some global properties of J-holomorphic curves. First, we note that using the almost complex structure J on P (and j on Σ), one can decompose

$$Tf = T^{(1,0)}f + T^{(0.1)}f \tag{3.4}$$

where $T^{(1,0)}f$ (resp. $T^{(0,1)}f$) is the J-linear (resp. anti-J-linear) part of Tf:

$$\begin{aligned} T^{(1,0)}f &:= \frac{1}{2}\left(Tf - J \circ Tf \circ j\right) \\ T^{(0,1)}f &:= \frac{1}{2}\left(Tf + J \circ Tf \circ j\right) \end{aligned} \tag{3.5}$$

Then f is J-holomorphic if and only if $T^{(0,1)}f \equiv 0$. We will also denote

$$\overline{\partial}_J f = T^{(0,1)}f \quad (\text{resp.} \quad \partial_J f = T^{(1,0)}f).$$

The followings are immediate consequences from the definition of J-holomorphic map and from the compatibility of J to ω.

Lemma 3.4. *Let $f : (\Sigma, j) \to (P, J)$ be a map. Then for the metric $g_J = \omega(\cdot, J\cdot)$ and the norm $|\cdot,\cdot|_J$*

1. $\frac{1}{2}|Tf|_J^2 = |\overline{\partial}_J f|_J^2 + |\partial_J f|_J^2$
2. $f^*\omega = (-|\overline{\partial}_J f|_J^2 + |\partial_J f|_J^2) dA$ *where dA is the area form on (Σ, j).*
3. *When $\overline{\partial}_J f = 0$, $f^*\omega = |\partial_J f|_J^2 dA \geq 0$ as a form*

Corollary 3.5. *For any smooth map $f : (\Sigma, j) \to (P, J)$, we have*

$$\frac{1}{2}\int_\Sigma |Tf|_J^2 dA \geq \int f^*\omega$$

and equality holds precisely when f is J-holomorphic.

Note that when Σ is a closed surface without boundary (resp. with boundary fixed or with free boundary on a fixed *Lagrangian* submanifold), then $\int f^*\omega$ is constant in a fixed homology class (resp. in a relative homology class). Therefore, *J-holomorphic curve minimizes the harmonic energy $\frac{1}{2}\int |Tf|_J^2$ in a fixed homology class.* This immediately gives rise to

Proposition 3.6. *When $f : (\Sigma, j) \to (P, J)$ is J-holomorphic,*

$$\text{Area}_{g_J} f = \frac{1}{2}\int |Tf|_J^2 = \int f^*\omega \tag{3.6}$$

Therefore $\text{Area}_{g_J} f$ *depends only on the homology represented by f. In particular, for any J-holomorphic map $f : (\Sigma, j) \to (P, J)$ representing a fixed homology class $A = [f]$,*

$$\text{Area}_{g_J}(f) = [\omega](A) \tag{3.7}$$

Corollary 3.7. *The image of J-holomorphic map is a minimal surface with respect to the metric $g_J = \omega(\cdot, J\cdot)$.*

Remark 3.8. In applications, one needs to vary the almost complex structure and to consider a family of J in a compact subset K of $\mathcal{J}_\omega$. However when we do estimates and convergence arguments of a sequence $\{f_i\}$ of maps , we have to use a metric g fixed among compatible metrics . Then (3.7) should be replaced by

$$\text{Area}_g(f) \leq C(A, K) \tag{3.8}$$

for any J-holomorphic map f and $J \in K \subset \mathcal{J}_\omega$, where $C(A, K)$ is a constant depending only on A and K. This uniform area estimate will be an important first step for obtaining uniform C^1-estimate and so for the compactness properties of J-holomorphic maps (for varying J).

§4. Outline of the proof of Nonsqueezing theorem

In this section, we give a detailed outline of the proof, given by Gromov [Gr], of the nonsqueezing theorem. The proof will be by contradiction.

Assume that there exists a symplectic embedding $\phi : B^{2n}(R) \to Z^{2n}(r)$. Since we assume that $R > r$, by considering a slightly smaller ball of radius R' with $r < R' < R$, we may assume that

$$B^{2n}(R) \subset \text{Int}(Z^{2n}(r)) \tag{4.1}$$

Then there exists some $\epsilon > 0$ such that ϕ extends to a symplectic embedding, still denoted by ϕ,

$$\phi : B^{2n}(R+\epsilon) \to Z^{2n}(r-\epsilon) \subset Z^{2n}(r).$$

We push forward J_0 from $B^{2n}(R+\epsilon)$ by ϕ to the image $\phi(B^{2n}(R+\epsilon)) \subset Z^{2n}(r-\epsilon)$. We consider an almost complex structure J_1 on $\mathbb{C}^n$ such that

$$J_1 = \begin{cases} \phi_*(J_0) & \text{on} \quad \phi(B^{2n}(R+\frac{\epsilon}{2})) \\ \quad J_0 & \text{on} \quad \mathbb{C}^n \ D^2(r-\epsilon) \times [-K+1, K-1]^{2(n-1)} \end{cases}$$

and J_1 is smoothly extended to the remaining region so that J_1 is still compatible to ω.

Exercise 5. Prove that this smooth extension is possible. (Hint: Use the polar decomposition and note that the decomposition is canonical.)

In the above, the constant $K > 0$ is chosen so that

$$\phi(B^{2n}(R+\epsilon)) \subset D^2(r-\epsilon) \times [-K+1, K-1]^{2(n-1)} \tag{4.2}$$

Here are the three steps to prove the nonsqueezing theorem.

Step I (Main step):

Prove that there exists a J_1-holomorphic map $f : (D^2, \partial D^2) \to (\mathbb{C}^n, \mathbb{C}^n \setminus \phi(B^{2n}(R+\epsilon)))$ such that

$$\int_C \omega_0 \leq \text{Area of the flat disk in } Z^{2n}(r) = \pi r^2 \tag{4.3}$$

where $C = \text{Image } f$, and

$$\phi(0) \in C \tag{4.4}$$

Step II:

Now consider the preimage of C, i.e.,

$$\phi^{-1}(C) \cap B^{2n}(R)$$

Since Image $(f|_{\partial D^2}) \subset \mathbb{C}^n \setminus \phi(B^{2n}(R+\epsilon))$, $\phi^{-1}(C) \cap B^{2n}(R)$ defines a *proper* surface in $B^{2n}(R)$ *passing through the origin.* Furthermore since $J_1|_{\phi(B^{2n}(R+\epsilon))} \equiv \phi_* J_0$ and C is J_1-holomorphic curve, $\phi^{-1}(C)$ is a J_0-holomorphic curve on $\mathbb{C}^n$ (and so a *minimal surface* with respect to the standard metric on $\mathbb{C}^n$). The following monotonicity formula (for minimal surface in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$) is well known in the minimal surface theory.(See e.g. [L], [HL])

Monotonicity formula. *Any proper minimal surface passing through the origin in $B^{2n}(R) \to \mathbb{C}^n$ has its area greater than or equal to the area of a flat disk ($= \pi R^2$) and equality holds iff the surface is a flat disk.*

This monotonicity immediately implies

$$\text{Area}(\phi^{-1}(C) \cap B^{2n}(R)) \geq \pi R^2 \tag{4.5}$$

because $\phi^{-1} \cap B^{2n}(R)$ is a proper J_0-holomorphic (and so minimal) surface passing through the origin.

Step III:

Get a contradiction: We apply (4.5), (3.6), the fact that ϕ is symplectic, change of variables, Lemma 3.4 (3) and (4.3) respectively to derive

$$\pi R^2 \leq \text{Area}\, \phi^{-1}(C) \cap B^{2n}(R) = \int_{\phi^{-1}(C) \cap B^{2n}(R)} \omega_0 = \int_{\phi^{-1}(C) \cap B^{2n}(R)} \phi^* \omega_0$$
$$= \int_{C \cap \phi(B^{2n}(R))} \omega_0 \leq \int_C \omega_0 \leq \pi r^2$$

which gives rise to a contradiction since we assume $R > r$.

Now, it remains to prove the statement in Step I. We first transform the existence statement on the J-holomorphic discs into a more tractable existence problem of J-holomorphic *sphere* on a *closed* symplectic manifold. By now, we are given an almost complex structure J_1 on $Z^{2n}(r) = D^2(r) \times \mathbb{C}^{n-1}$ such that J_1 is standard on $\mathbb{C}^n \setminus D^2(r - \epsilon) \times [-K+1, K-1]^{2(n-1)}$. In particular, J_1 is standard near the boundary of the domain $D^2(r - \frac{\epsilon}{2}) \times [-K, K]^{2(n-1)} =: D_{r,\epsilon,K}$. Therefore, we can push forward J_1 on $D_{r,\epsilon,K}$ to

$$D^2(r - \frac{\epsilon}{2}) \times T^{2(n-1)}(K)$$

which we still denote by J_1, where $T^{2(n-1)}(K)$ is the torus obtained by identifying $-K$ with K in $[-K, K]$, i.e.,

$$T^{2(n-1)}(K) = S^1_K \times \cdots \times S^1_K \quad 2(n-1) \text{ times}$$

Note that since $\phi(B^{2n}(r + \epsilon)) \subset D_{r,\epsilon,K}$, we may consider $\phi(B^{2n}(r + \epsilon))$ as a subset of

$$D^2(r - \frac{\epsilon}{2}) \times T^{2(n-1)}(K).$$

The corresponding almost complex structure J_1 on this space is still standard *near the boundary of the space*.

Now, we embed $D^2(r - \frac{\epsilon}{2})$ into $S^2(\frac{r}{2})$ by an area preserving map $\psi : D^2(r - \frac{\epsilon}{2}) \to S^2(\frac{r}{2})$ and then embed $D^2(r - \frac{\epsilon}{2}) \times T^{2(n-1)}(K)$ into

$$P_{r,K} := S^2(\frac{r}{2}) \times T^{2(n-1)}(K)$$

by the symplectic map $\psi \times id : D^2(r - \frac{\epsilon}{2}) \times T^{2(n-1)}(K) \to P_{r,K}$. We denote by $\omega_{r,K} = \omega_1^r \oplus \omega_2^K$ the product symplectic structure on $P_{r,K}$ where ω_1^r and ω_2^K are the standard symplectic structure on $S^2(\frac{r}{2})$ and $T^{2(n-1)}(K)$ respectively. We now extend the structure $(\psi \times id)_* J_1$ on $(\psi \times id)(D^2(r - \frac{\epsilon}{2}) \times T^{2(n-1)}(K))$ to the whole $P_{r,K} = S^2(\frac{r}{2}) \times T^{2(n-1)}(K)$ so that the extension, denoted by $\widetilde{J}_1$ is compatible to $\omega_{r,K}$. Now note that the second homotopy group $\pi_2(P_{r,K}) \simeq \mathbb{Z}$ and the homotopy class $[S^2(\frac{r}{2}) \times \{pt\}]$ generates π_2. Denote this homotopy class by A and let $p_0 \in P_{r,K}$ be the point corresponding to $\phi(0)$ in $Z^{2n}(r)$.

Assertion: To finish Step I, it is enough to find a $\widetilde{J}_1$-holomorphic sphere $\widetilde{C}$ with $[\widetilde{C}] = A \in \pi_2(P_{r,K})$ and with $p_0 \in \widetilde{C}$.

Proof of Assertion. Suppose that there exists a $\widetilde{J}_1$-holomorphic sphere $\widetilde{C}$ and let $u : S^2 \to P_{r,K}$ be the map representing $\widetilde{C}$, i.e., $\widetilde{C} = \text{Image}(u)$. Then since $[\widetilde{C}] = A$ and from (3.6), we have

$$\int_{\widetilde{C}} \omega_{r,K} = \pi r^2 \tag{4.6}$$

Since $[\widetilde{C}] = A = [S^2(\frac{r}{2}) \times \{pt\}]$, it is easy to prove that the composition $\pi \circ u : S^2 \to S^2(\frac{r}{2})$ is surjective (Prove this. **Exercise 7**). Now the curve C required in Step I can be chosen to be

$$C = (\psi \times id)^{-1}(\widetilde{C}) \subset D^2(r - \frac{\epsilon}{2}) \times T^{2(n-1)}(K)$$

regarded as a subset of $Z^{2n}(r - \frac{\epsilon}{2}) \subset Z^{2n}(r)$. Obviously, C is a proper surface in $Z^{2n}(r - \frac{\epsilon}{2})$ and so $C \cap \phi(B^{2n}(r + \epsilon))$ defines a proper surface in $\phi(B^{2n}(r + \epsilon))$ since we assumed $\phi(B^{2n}(r + \epsilon)) \subset Z^{2n}(r - \frac{\epsilon}{2})$ in the beginning of this section. Furthermore, since $\psi \times id$ is a symplectic map and from the definition of $\widetilde{J}_1$, we have

$$\begin{aligned}
\int_C \omega_0 &= \int_C (\psi \times id)^* \omega_{r,K} \\
&= \int_{(\psi \times id)^{-1}(\widetilde{C})} (\psi \times id)^* \omega_{r,K} \\
&= \int_{\widetilde{C} \cap (\psi \times id)(D^2(r - \frac{\epsilon}{2}) \times T^{2(n-1)}(K))} \omega_{r,K} \\
&\leq \int_{\widetilde{C}} \omega_{r,K} = \pi r^2
\end{aligned}$$

which finish the proof of (4.3). Here the last inequality follows from Lemma 3.4(3). (4.4) follows from that $p_0 \in \widetilde{C}$. □

§5. Fredholm set-up and the existence scheme

Now we have reduced the proof of the non-squeezing theorem to the following general existence theorem of J_1-holomorphic spheres.

Theorem 5.1. *Let $P_{r,K} = S^2(\frac{r}{2}) \times T^{2(n-1)}(K)$ with the symplectic form $\omega_{r,K} = \omega_1^r \oplus \omega_2^K$, J_β be any given compatible almost complex structure and let $p_0 = (x_0, q_0) \in P_{r,K}$ be a given point in $P_{r,K}$. Let $A = [S^2(\frac{r}{2}) \times \{pt\}]$ be the generator of $\pi_2(P_{r,K})$. Then there exist a J_β-holomorphic sphere $u : S^2 \to P_{r,K}$ with $[u] = A$ and $p_0 \in$ Image(u).*

Of course, we apply this theorem to $J_\beta = \widetilde{J}_1$ in the previous section for the finish of Step I. The proof of this theorem requires all the basic ingredients in the Gromov pseudo-holomorphic curve theory and so we will set up the required Fredholm set-up in a completely general context. Let (P, ω) be an arbitrary compact (or at least *tame* in the sense of Gromov [Gr]) symplectic manifold. Again we denote by $\mathcal{J}_\omega$ the set of almost complex structures compatible to ω. Denote by

$$\mathcal{F} = \mathcal{F}^{k,p} = W^{k,p}(S^2, P) \tag{5.1}$$

the set of Sobolev $W^{k,p}$-maps from S^2 to P, where

$$k - \frac{2}{p} > 0, \quad p > 2$$

so that $\mathcal{F} \subset C^0(S^2, P)$. For each $f \in \mathcal{F}^{k,p}$, we decompose

$$Tf = \overline{\partial}_J f + \partial_J f$$

where we recall

$$\overline{\partial}_J f = \frac{Tf + J \circ Tf \circ j}{2}$$
$$\partial_J f = \frac{Tf - J \circ Tf \circ j}{2}.$$

Recall that $\overline{\partial}_J f$ is anti-J-linear and $\partial_J f$ is J-linear maps. We denote by

$$H_{(f,J)} = \mathrm{Hom}_J^{(1,0)}(TS^2, f^*TP)$$
$$\overline{H}_{(f,J)} = \mathrm{Hom}_J^{(0,1)}(TS^2, f^*TP)$$

the bundle of J-linear maps and anti-J-linear maps over $f : S^2 \to P$, respectively. We are particularly interested in $\overline{H}_{(f,J)}$ and its sections. We would like to note that

$$\mathrm{rank}\, \overline{H}_{(f,J)} = \mathrm{rank}\, f^*(TP) = 2n,$$

which makes the equation $\overline{\partial}_J f = 0$ well posed. Denote by

$$\overline{\mathcal{H}}_{(f,J)} = \overline{\mathcal{H}}^{k-1,p}_{(f,J)} = W^{k-1,p}(\overline{H}_{(f,J)}) \tag{5.2}$$

the set of $W^{k-1,p}$-section of the bundle $\overline{H}_{(f,J)}$ and form an (infinite-dimensional) vector bundle $\overline{\mathcal{H}}$ over $\mathcal{F} \times \mathcal{J}_\omega$, where

$$\overline{\mathcal{H}} = \bigcup_{(f,J)} \overline{\mathcal{H}}_{(f,J)}.$$

This is a vector bundle modeled by a Banach space $\overline{\mathcal{H}}_{(f,J)}$. Then the assignment

$$(f, J) \mapsto \overline{\partial}_J f$$

defines a smooth section of the vector bundle $\overline{\mathcal{H}}$ over $\mathcal{F} \times \mathcal{J}_\omega$. We denote this section by $\overline{\partial}$. We will be interested in the pair (f, J) that is a zero of the section $\overline{\partial}$, i.e., that satisfies

$$\overline{\partial}_J f = 0.$$

Denote the zero set of $\overline{\partial}$ by

$$\mathcal{M}_A = \{(f, J) \in \mathcal{F} \times \mathcal{J}_\omega \mid \overline{\partial}_J f = 0, [f] = A, f \text{ satisfying } (H) \text{ below}\}$$

For the transversality theorem below, it is important to restrict to f's satisfying the following hypothesis:

(H). *There exists a point* $z \in S^2$ *such that* $Tf(z) \neq 0$ *and* $\#(f^{-1}(z)) = 1$.

We call any J-holomorphic curve satisfying (H) a *simple curve* and otherwise a *multiple curve*. (See [M1] for a detailed discussion on the structure of curves satisfying (H).)

Proposition 5.2 [M1]. $\mathcal{M}_A \subset \mathcal{F} \times \mathcal{J}_\omega$ *is a smooth submanifold and the projection map* $\pi_2 : \mathcal{M}_A \to \mathcal{J}_\omega$ *is a Fredholm map with index* $2(c_1(A) + n)$ *where* c_1 *is the first Chern class of the vector bundle* TP *considered as a complex vector bundle.*

Now, the Sard-Smale theorem [S] immediately implies

Corollary 5.3. *There exist a dense subset* $\mathcal{J}_{\omega,A} \subset \mathcal{J}_\omega$ *such that the set*

$$\mathcal{M}_A(J) = \{f \in \mathcal{F} \mid \overline{\partial}_J f = 0, f \text{ satisfying (H)}\}$$

becomes a finite dimensional smooth manifold of dimension $2(c_1(A) + n)$. *We call* $J \in \mathcal{J}_{\omega,A}$ *A-regular.*

Remark 5.4. One can also prove the parametrized version of Proposition 5.2 and Corollary 5.3. Our main interest is the case of one parameter family of almost complex structure J in $\mathcal{J}_\omega$.

Now, we go back to the proof of Theorem 5.1. To prove this, we will use a version of the well known *continuity method* in PDE. We start from the standard (integrable) product structure J_α on $S^2(\frac{r}{2}) \times T^{2(n-1)}(K)$.

Lemma 5.5. *The product structure* J_α *on* $S^2(\frac{r}{2}) \times T^{2(n-1)}(K)$ *is A-regular where* $A = [S^2(\frac{r}{2}) \times \{pt\}]$, *and* $\mathcal{M}_A(J_\alpha)$ *becomes*

$$\begin{aligned}\mathcal{M}_A(J_\alpha) = \{f : S^2 \to P_{r,K} \mid f(z) = (f_1(z), q),\ \ q \in T^{2(n-1)}(K),\\ f_1 : S^2 \to S^2(\frac{r}{2}) \text{ is a bi-holomorphism } \}\end{aligned}$$

Proof. **Exercise 6**.

Note that $c_1(A) = 2$ and so $\dim \mathcal{M}_A(J_\alpha) = 2(2+n) = 2n+4$ which is consistent with the explicit description of $\mathcal{M}_A(J_\alpha)$ in the above lemma. Recall that we are interested in finding a J_β-holomorphic sphere passing through a given point p_0. So we consider the following evaluation map

$$\mathrm{ev}_A : \mathcal{M}_A(J) \times_G S^2 \to P, \quad \mathrm{ev}_A(f, z) = f(z)$$

where $\mathcal{M}_A(J_\alpha) \times_G S^2 := \mathcal{M}_A(J) \times S^2/G$ in which the action of G (= the set of automorphisms of S^2 on $\mathcal{M}_A(J) \times S^2$) is given by

$$g : (f, z) \mapsto (f \circ g^{-1}, g(z)).$$

This action is free because we assume that f satisfies (H) and so $\mathcal{M}_A(J) \times_G S^2$ is a manifold where $\mathcal{M}_A(J)$ is so. The dimension for this is given by

$$2(c_1(A) + n) + 2 - 6 = 2c_1(A) + 2n - 4.$$

In the case of the product structure J_α in Lemma 5.5 where $c_1(A) = 2$, this dimension becomes $2n$. In fact, we have

Lemma 5.6. *For the product structure J_α, $\mathcal{M}_A(J)\times_G S^2$ is a compact $2n$-manifold and the evaluation map*

$$\mathrm{ev}_A : \mathcal{M}_A(J) \times_G S^2 \to P_{r,K}$$

becomes a diffeomorphism. In particular, it has non-zero $\mathbb{Z}_2$-degree.

Proof. We leave the proof to reader or refer to [M2] for a relevant argument to prove this lemma.

Now, we consider the given J_β in Theorem 5.1. In general, J_β may not be A-regular and so we choose a sequence of A-regular J_i's such that

$$J_i \to J_\beta \quad as \quad i \to \infty$$

in the C^∞-topology. Now, for each fixed i, we consider a path $\overline{J} = \{J_t\}_{0\leq t\leq 1}$ such that $J_0 = J_\alpha$ and $J_1 = J_i$ and consider the parametrized evaluation map

$$\mathrm{Ev}_A : \mathcal{M}_A(\overline{J}) \times_G S^2 \to P_{r,K} \times [0,1] \quad \mathcal{M}_A(\overline{J}) := \bigcup_{t\in[0,1]} \mathcal{M}_A(J_t)$$

defined by

$$\mathrm{Ev}_A(f_t, J_t, z) = (f_t(z), t).$$

If one can prove that $\mathcal{M}_A(\overline{J})\times_G S^2$ is a compact manifold, it will provide a compact cobordism between $\mathcal{M}_A(J_\alpha) \times_G S^2$ and $\mathcal{M}_A(J_i) \times_G S^2$ and so the evaluation map

$$\mathrm{ev}_A^0 : \mathcal{M}_A(J_\alpha) \times_G S^2 \to P_{r,K} \times \{0\}$$

and

$$\mathrm{ev}_A^1 : \mathcal{M}_A(J_i) \times_G S^2 \to P_{r,K} \times \{1\}$$

must have the same $\mathbb{Z}_2$-degree. Therefore since ev_A^0 has $\mathbb{Z}_2$-degree 1 from Lemma 5.6,

$$\mathrm{ev}_A : \mathcal{M}_A(J_i) \times_G S^2 \to P_{r,K}$$

must have non-zero $\mathbb{Z}_2$-degree and in particular is surjective. Hence there exists a J_i-holomorphic curve passing through p_0. Let us denote such a J_i-holomorphic curve by f_i. Now, *if one could prove the uniform estimate for the derivative Tf_i*

$$\max_{z\in S^2} |Tf_i(z)| \leq C \tag{5.3}$$

for all i after reparametrization if necessary, one could pass to the limit to find a J_β-holomorphic sphere f_∞ such that $p_0 \in \mathrm{Image}\, f_\beta$ which would finish the proof of Theorem 5.1 and so the nonsqueezing theorem.

Both of the above two compactness statement will be a consequence of *Gromov's weak-compactness theorem* which we will describe in the next section. To apply Gromov's compactness theorem to our situation, the uniform area bound (3.8) and the fact that the homotopy class $A = [S^2(\frac{r}{2}) \times \{pt\}]$ is *simple* will be used.

Definition 5.7. A homotopy class A is called *J-simple* if there is no decomposition of A into $A = \sum_{j=1}^N A_j$ such that A_j allows a non-trivial J-holomorphic curve. If this holds for any $J \in \mathcal{J}_\omega$, then we call A *simple.*

With this definitioin, one can easily see that the homotopy class $A = [S^2 \times \{pt\}]$ is simple: If there could be such a decomposition, then we would have

$$[\omega](A) = \sum_{j=1}^N [\omega](A_j).$$

But if A_j allows any J-holomorphic curve, then

$$[\omega](A_j) > 0 \quad \text{and so} \quad [\omega](A_j) \geq \pi r^2.$$

On the other hand, we have $[\omega](A) = \pi r^2$ and such a decomposition is not possible.

§6. Compactness

The main goal of this section is to prove the two compactness statement left out in the last section. We first consider

Proposition 6.1. *Let $P = S^2(\frac{r}{2}) \times T^{2(n-1)}(K)$ and $A \in \pi_2(P)$ be as before. Then for any path $\overline{J} = \{J_t\}_{0 \leq t \leq 1} \subset \mathcal{J}_\omega$, $\mathcal{M}_A(\overline{J}) \times_G S^2$ is compact. Furthermore, if we fix $J_0 = J_\alpha$, $J_1 = J_\gamma$ that are A-regular, there exists a smooth path $\overline{J}$ such that $\mathcal{M}_A(\overline{J}) \times_G S^2$ becomes a compact smooth manifold.*

To prove this proposition, we need the fundamental Gromov's compactness theorem [Gr]. We refer to [PW] for an elegant complete proof of this compactness theorem. We first recall the notion of *J-cusp-curves* (We refer to [PW] for a more precise definition).

Definition 6.2. A J-cusp-curve in P is a finite union $C = \{C_\ell\}$ of J-holomorphic curve where C_ℓ is the image of a J-holomorphic map $f_\ell : S^2 \to P$. We call a J-cusp-curve *connected* if the union $\cup_\ell C_\ell$ is connected as a set.

Gromov's weak compactness theorem [Gr]. *Let Let $J_\gamma \to J_\infty$ be a convergent sequence in $\mathcal{J}_\omega$. Then for any sequence of J_γ-holomorphic maps $f_\gamma : S^2 \to P$ with the uniform area bound*

$$\text{Area}(f_\gamma) < C,$$

there exists a subsequence of f_γ such that the unparametrized curve $C_\gamma = \text{Image}\, f_\gamma$ weakly-converges to a cusp curve $C_\infty = \{C_{\infty,l}\}_{1 \leq l \leq N}$ so that

(1) $\text{Area}(C_\infty) = \overline{\lim}_{\gamma \to \infty} \text{Area}(C_\infty) = \sum_{l=1}^N \text{Area}(C_{\infty,l})$
(2) $[C_\infty] = \sum_{l=1}^N [C_{\infty,l}]$ *in* $\pi_2(P)$
(3) C_∞ *is a connected cusp curve.*

Furthermore, if $N = 1$, i.e., C_∞ is a genuine curve, then the subsequence f_γ C^∞-converges to a limit f_∞ after reparametrization if necessary.

Now, we are ready to prove Proposition 6.1

Proof of Proposition 6.1. Since $\mathcal{M}_A(\overline{J}) \times_G S^2$ is a S^2-bundle over $\mathcal{M}_A(\overline{J})/G$, it is enough to prove that $\mathcal{M}_A(\overline{J})/G$ is compact. Let (f_i, J_{t_i}) be a sequence in $\mathcal{M}_A(\overline{J})$. Then we have the uniform area bound

$$\text{Area}(f_i) \leq C(A)$$

from (3.8). Then the above compactness theorem implies that there exists a subsequence of f_i still denoted by f_i such that the unparametrized curve $C_i = \text{Image}\, f_i$ converges to a cusp-curve $C_\infty = \sum_\ell^N C_{\infty,\ell}$. By choosing a subsequence, we may assume that $t_i \to t_\infty$. However, since A is simple, N cannot be bigger than 1 by the argument in the end of the previous section and so the subsequence f_i converges to a J_{t_∞}-holomorphic curve after reparametrization, which finishes the proof of the fact that $\mathcal{M}_A(\overline{J})/G$ is compact. The last statement will follow by the transversality theorem if we choose a path $\overline{J}$ that is transverse to the projection $\pi_2 : \mathcal{M}_A \to \mathcal{J}_\omega$ in Proposition 5.2. □

We recall where we were in the end of section 5: we are given a sequence of A-regular J_i such that $J_i \to J_\beta$ where J_β as in Theorem 5.1.

By applying Proposition 6.1 and the existence scheme used in the end of section 5 for each J_i, we are given a sequence f_i of J_i-holomorphic sphere passing through the point p_0. Again using the area bound (3.8) and the simpleness of A, the above Gromov's compactness theorem guarantees the existence of the limit f_∞ (after reparametrization) such that f_∞ is J_β-holomorphic and $p_0 \in \text{Image}\, f_\infty$. This finally finishes the proof of the nonsqueezing theorem.

§7. Epilogue: Symplectic rigidity.

In the middle of 1980's, Eliashberg proved the following theorem, which first indicated the existence of *symplectic topology* that is supposed to be finer than *differential topology.*

Theorem 7.1 [Eliashberg]. *The subgroup $\mathcal{D}_\omega(P)$ of symplectic diffeomorphisms is C^0-closed in Diff(P). More precisely, if a sequence ϕ_j of symplectic diffeomorphism has the C^0-limit ϕ_∞ that is differentiable, then ϕ_∞ is a (C^1)-symplectic diffeomorphism.*

Eliashberg's proof [E] relies on a structure theorem on the *wave fronts* of certain Legendrian submanifolds. The complete details of the proof of this structure theorem has not appeared yet in the literature. However, Gromov's non-squeezing theorem can replace Eliashberg's argument to give another proof of Theorem 7.1. In fact, the existence of *any symplectic capacity* function on the set of open set in $\mathbb{C}^n$ will provide a relatively straightforward proof of Theorem 7.1, using Eliashberg's argument in [E]. We refer [EH2, HZ] for details of such a proof.

In the remaining section, we will give a definition of one such symplectic capacity using the non-squeezing theorem. Again, we refer readers to the book [HZ] for a very good exposition on the symplectic rigidity theory from the point of Hamiltonian dynamics and the critical point theory.

Definition 7.2 [Gromov radius]. For any symplectic manifold (P, ω) we define

$$\underline{c}(P, \omega) = \sup\{\pi r^2 \mid \exists \text{ a symplectic embedding } \ \phi : (B(r), \omega_0) \to (P, \omega)\}$$

and call it the *Gromov radius* of (P, ω).

The followings are the main properties of $\underline{c}$.

Proposition 7.3.

(1) *Monotonicity: $\underline{c}(P, \omega) \le \underline{c}(\widetilde{P}, \widetilde{\omega})$ if there exists a symplectic embedding $\phi : (P, \omega) \to (\widetilde{P}, \widetilde{\omega})$.*
(2) *Conformality: $\underline{c}(P, \alpha\omega) = |\alpha| \underline{c}(P, \omega)$ for all $\alpha \in \mathbb{R}$ $\alpha \neq 0$.*
(3) *Nontriviality: $\underline{c}(B^{2n}(1), \omega_0) = \pi = \underline{c}(Z^{2n}(1), \omega_0)$*

Proof. (1) and (2) are immediate consequences from the definition of $\underline{c}$ and (3) is an easy consequence of the non-squeezing theorem. (**Exercise 7**) □

In fact, any assignment $c : (P, \omega) \mapsto c(P, \omega)$ that associates with every symplectic manifold (P, ω) a nonnegative real number or ∞ satisfying the Axiom (1), (2) and (3) is called a *symplectic capacity*. By now, there are several different constructions of the existence of different symplectic capacities. We refer to [HZ] for a detailed exposition on the symplectic capacity theory. In this point of view, the non-squeezing theorem provided the first construction of such symplectic capacity.

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