4 - Gromov compactness

Note Title 8/6/2009

Goal: $\{u : S^2 \to S^2 \times T \mid \bar\partial_J u = 0 ,\ \underbrace{u_*[S^2] = [S^2 \times pt]}_{\Rightarrow E(u) = \frac{1}{2}\|du\|^2_{L^2} = \pi}\} / Aut(S^2)$ is compact

false: $\{u : \Sigma \to M \mid \bar\partial_J u = 0 ,\ E(u) \leq C\}$ is compact

Thm: $\{ \text{———————} \ \|du\|_{L^p} \leq C\}$ is compact for $p > 2$

Proof: $W^{k+1,p}(\Sigma) \hookrightarrow \mathcal{C}^k(\Sigma)$ compact and

$\boxed{\text{elliptic estimates for } \bar\partial_J}$

Thm: $u \in W^{1,p}(\Sigma, M),\ \bar\partial_J u = 0 \underset{p>2}{\Rightarrow} u \in \mathcal{C}^\infty(\Sigma, M),\ \|du\|_{W^{k,p}} \leq C_k(\|du\|_{L^p})$

continuous function

Proof of Thm uses

(0) local coordinates $\Omega \subset \mathbb{C}$, $u : \Omega \to \mathbb{R}^{2n}$, $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n \times 2n}$

(i) Ellipticity of $\Delta = \partial_s^2 + \partial_t^2$: $K \subset \Omega$ compact, $1 < q < \infty$

$u \in W^{k,q}(\Omega),\ \Delta u \in W^{k-1,q}(\Omega) \Rightarrow u \in W^{k+1,q}(K)$ and

$$\|u\|_{W^{k+1,q}(K)} \leq C\left(\|\Delta u\|_{W^{k-1,q}(\Omega)} + \|u\|_{L^q(\Omega)}\right)$$

(ii) $(\partial_s - J(u)\partial_t)\ \underbrace{\bar\partial_J u}_{= 0} = \Delta u + (\underbrace{\nabla_{\partial_s u}}_{L^2} J)\underbrace{\partial_t u}_{L^2} - (\underbrace{\nabla_{\partial_t u}}_{L^2} J)\underbrace{\partial_s u}_{L^2}$

Note: $u \in W^{1,2},\ \bar\partial_J u = 0 \Rightarrow \Delta u \in L^1 \to$ (i) doesn't apply

$(p > 2,\ \frac{1}{p} + \frac{1}{p'} = 1 \Rightarrow p' < 2)$ $\quad \in (W^{1,p})^* = W^{-1,p'} \Rightarrow u \in W^{1,p'}$ ‖ no mileage gained

(iii) elliptic bootstrapping

$$u \in W^{1,p} \Rightarrow \Delta u = -\overbrace{(\nabla J)(u)}^{\text{linear map } \mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}}(\partial_s u, \partial_t u) + \overbrace{(\nabla J)(u)}(\partial_t u, \partial_s u)$$

$$L^p \cdot L^p \qquad L^p \cdot L^p \subset L^{p/2} \qquad \tfrac{1}{p}+\tfrac{1}{p}=\tfrac{2}{p}$$

$W^{2,p} \Rightarrow$ $\quad W^{1,p}\cdot W^{1,p} \subset W^{1,p}$ for $p>2$

$W^{k,p} \Rightarrow$ $\quad W^{k-1,p}\cdot W^{k-1,p} \subset W^{k-1,p}$ for $(k-1)p>2$

$$\Rightarrow u \in W^{2,p/2}, \quad \|u\|_{W^{2,p/2}} \le C\left(\|du\|^2_{L^p} + \|u\|_{L^{p/2}}\right)$$

if $p/2<2$: $\quad \hookrightarrow W^{1,q} \qquad \ge \frac{1}{c}\|u\|_{W^{1,q}} \qquad q=\frac{2p}{p-2} > p$ (! improvement !)

iterate to get to $W^{1,q>4} \Rightarrow W^{2,q/2=:r>2}$, then further iterate

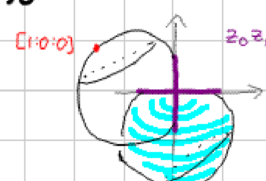
to get $u \in W^{k+1,r}$, $\|u\|_{W^{k+1,r}} \le C_k\left(\|du\|^2_{W^{k-1,r}} + \|u\|_{L^r}\right) \quad \forall k \in \mathbb{N}$

Bubbling Analysis

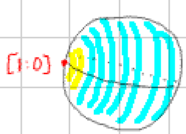
moduli spaces $\{\bar\partial_J u = 0, E(u) = \langle A,[\omega]\rangle\}/\mathrm{Aut} \cong \{C \subset M \text{ J-hol}, \int_C \omega = \langle A,[\omega]\rangle\}$

are generally noncompact

Ex: $C_\varepsilon = \{z_0 z_1 = \varepsilon z_2^2\} \subset \mathbb{CP}^2 \xrightarrow[\varepsilon\to 0]{} \{[z_0:0:z_2]\} \cup \{[0:z_1:z_2]\}$

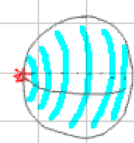


2 spheres with transverse intersection at $[0:0:1]$



$u_\varepsilon : \mathbb{CP}^1 \to \mathbb{CP}^2$
$[x:y] \mapsto [\varepsilon x^2 : y^2 : xy]$

$\xrightarrow[\varepsilon\to 0]{}$ except at $[1:0]$ where $|du_\varepsilon| \to \infty$



$u_0[x:y] = [0:y^2:xy] = [0:y:x]$

remove singularity by $u_0[1:0] = [0:0:1]$

(near $[1:0] \mapsto [1:0:0]$
$u_\varepsilon : [1:y] \mapsto [1 : \frac{1}{\varepsilon}y^2 : \frac{1}{\varepsilon}y]$
$\frac{du_\varepsilon}{dy}(y) = (\frac{2}{\varepsilon}y, \frac{1}{\varepsilon})$)

Even worse, $v_\varepsilon : \mathbb{CP}^1 \to \mathbb{CP}^2,\ [x:y] \mapsto [\sqrt{\varepsilon}x^2 : \sqrt{\varepsilon}y^2 : xy]$ parametrizes C_ε

but converges to $v_0[x:y] = [0:0:1]$ except for $|dv_\varepsilon|$-blowup at $[1:0]$ and $[0:1]$.

One blowup can be avoided by reparametrization $v_\varepsilon \circ \varphi_\varepsilon = u_\varepsilon$ $\left(\varphi_\varepsilon[x:y] = [\sqrt{\varepsilon}x : \tfrac{y}{\sqrt{\varepsilon}}]\right)$

$\left(\text{or } v_\varepsilon \circ (\varphi_\varepsilon[x:y] = [\tfrac{x}{\sqrt{\varepsilon}} : \sqrt{\varepsilon}y]) = [x^2 : \varepsilon y^2 : xy] \xrightarrow[\varepsilon \to 0]{} [x:0:y]\right)$ but not both.

<u>Gromov compactness</u>: Any sequence $u_i : \Sigma \to M$, $\bar\partial_{J_i} u_i = 0$

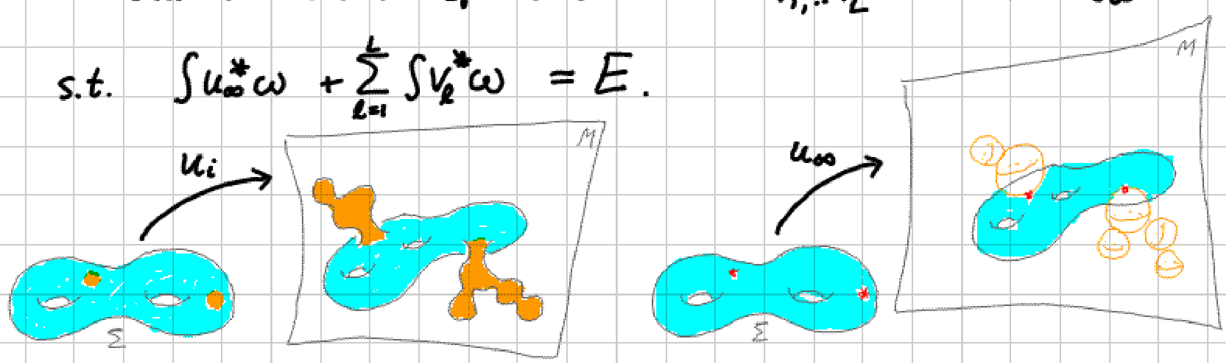
! j on Σ fixed !

with fixed energy $\sup_{i \in \mathbb{N}} E(u_i) = \int_\Sigma u_i^* \omega = E$ and $J_i \xrightarrow{C^\infty} J_\infty$

has a subsequence that "Gromov-converges" to a J_∞-hol. map $u_\infty : \Sigma \to M$

and "stable trees of sphere bubbles" $v_1, \dots v_L : S^2 \to M$ J_∞-hol.

s.t. $\int u_\infty^* \omega + \sum_{\ell=1}^{L} \int v_\ell^* \omega = E$.



More precisely: For a subsequence (again denoted $(u_i)_{i \in \mathbb{N}}$)

(1.) The energy densities $|du_i|^2 : \Sigma \to \mathbb{R}$ converge as measure

$$e_i = |du_i|^2 \xrightarrow[i \to \infty]{} \underset{\underset{\mathcal{C}^0(\Sigma,[0,\infty))}{\cap}}{e_\infty} + \sum_{k=1}^{N} \underset{\underset{\hbar > 0}{\vee\!/}}{E_k}\, \delta_{p_k} \qquad \text{for } p_1 \dots p_N \in \Sigma$$

(2.) For every blow-up point $p \in \Sigma$ (i.e. $\exists\, p_i \xrightarrow[i\to\infty]{} p$, $|du_i(p_i)| \to \infty$)

in local coordinates $0 \in \Omega \subset \mathbb{C}$ $\quad \exists\, z_i \to 0$, $|du_i(z_i)| = R_i \to \infty$

s.t. the rescaled maps $v_i(z) = u_i(z_i + z/R_i) \; : B_{\varepsilon_i R_i} \to M$, $\quad B_{\varepsilon_i R_i} \subset \mathbb{C}$, $\quad B_{\varepsilon_i}(z_i) \subset \Omega$

$\downarrow \mathcal{C}^\infty_{loc}$

converge to a "sphere" $v_\infty : \mathbb{C} \to M$

(by removal of singularity) $\tilde{v}_\infty : \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \to M$ nonconstant

(Note this does not yet capture the full energy / full tree of spheres.)

<u>Cor</u>: For nonsqueezing, the moduli spaces of J-hol. $u : S^2 \to S^2 \times T$

with energy $E(u) = \int_{S^2} u^*\omega = \langle [S^2 \times pt], [\omega_0^{S^2} \times K\omega_0^T] \rangle = \pi$ are compact

modulo $Aut(S^2)$.

$\to$ any bubble $\tilde{v}_\infty : S^2 \to S^2 \times T$ has energy > 0, $\in \pi\mathbb{Z} = \langle \pi_2(S^2 \times T), [\omega_0 \times K\omega_0] \rangle$

$\Rightarrow\ \geq \pi$

$\Rightarrow$ $u_i \xrightarrow[\mathcal{C}^\infty(S^2)]{} u_\infty$ $\quad$ or $\quad$ $\underbrace{u_i \xrightarrow[\mathcal{C}^\infty(S^2 \setminus \{p\})]{} u_\infty \equiv \text{const} \ \& \text{ one bubble } \tilde{v}_\infty}$

rescaling gives $u_i \circ \varphi_i \xrightarrow[\mathcal{C}^\infty(S^2)]{} \tilde{v}_\infty$

Tool kit for bubbling analysis

• **Removable singularity**: $u : B\setminus\{0\} = \{z\in\mathbb{C} \mid 0<|z|<1\} \longrightarrow M$

$$\bar\partial_J u = 0 \,,\quad \int_{B\setminus\{0\}} |du|^2 < \infty \quad \Rightarrow \quad \lim_{z\to 0} u(z) =: \tilde u(0) \text{ exists}$$

and extends u to $\tilde u \in \mathcal{C}^\infty(B,M)$, $\bar\partial_J \tilde u = 0$.

• **Hofer trick**: (X,d) complete metric space, $f \in \mathcal{C}^0(X,[0,\infty))$

$$\forall p\in X,\ \varepsilon>0 \quad \exists z\in B_{2\varepsilon}(p),\ 0<\varepsilon'\le\varepsilon : \quad \sup_{B_{\varepsilon'}(z)} f \le 2f(z)\,,\ \varepsilon' f(z) \ge \varepsilon f(p)$$

• **mean value inequality**: $e\in\mathcal{C}^2(\mathbb{R}^2,[0,\infty))$

$$\left.\begin{array}{l} \Delta e \ \ge -A_0 - a e^2 \\ \displaystyle\int_{B_r(0)} e < \underbrace{\hbar}_{\hbar(a)>0} \end{array}\right\} \Rightarrow \quad r^2 e(0) \le C\Big(\int_{B_r(0)} e + A_0 r^4 \Big)$$

• $\partial_s u + J(u)\partial_t u = 0 \quad \Rightarrow \quad e = |du|^2 = 2|\partial_s u|^2$ satisfies

$$(\partial_s^2+\partial_t^2)e = 4|\partial_s\partial_s u|^2 + 4|\partial_t\partial_s u|^2 + 4 g_J(\Delta\partial_s u, \partial_s u)$$

$$\ge \dots \ge -4a|\partial_s u|^4 = -a e^2$$

Ingredients for proof of ①,② (backwards) with application

• Removable singularity: $u : B\setminus\{0\} = \{z\in\mathbb{C} \mid 0<|z|<1\} \longrightarrow M$

$\bar\partial_J u = 0$, $\int_{B\setminus\{0\}} |du|^2 < \infty \quad \Rightarrow \lim_{z\to 0} u(z) =: \tilde u(0)$ exists

and extends u to $\tilde u \in \mathcal{C}^\infty(B,M)$, $\bar\partial_J \tilde u = 0$.

• Hofer trick: (X,d) complete metric space, $f\in\mathcal{C}^0(X,[0,\infty))$
$\overset{\shortparallel}{\Sigma}$ with metric $\qquad \overset{\shortparallel}{|du_i|}$

$\forall p\in X,\ \varepsilon>0\ \exists z\in B_{2\varepsilon}(p),\ 0<\varepsilon'\le\varepsilon :\ \sup_{B_\varepsilon(z)} f \le 2f(z)\ ,\ \varepsilon' f(z) \ge \varepsilon f(p)$

$\overset{\shortparallel}{p_i\to p}$ $\quad \overset{\shortparallel}{|du_i(p_i)|^{-1/2}} \to 0$ $\quad \overset{\shortparallel}{z_i\to p}$ $\quad \overset{\shortparallel}{\varepsilon_i}$

$\sup_{B_{\varepsilon_i}(z_i)} |du_i| \le 2\underbrace{|du_i(z_i)|}_{=2R_i}$, $\varepsilon_i R_i \ge |du_i(p_i)|^{1/2} \to\infty$

$\Rightarrow |dv_i(0)| = 1$, $\sup_{B_{\varepsilon_i R_i}} |dv_i| \le 2 \quad \Rightarrow$ subsequence $v_i \longrightarrow v_\infty$ nonconstant

• mean value inequality: $e\in\mathcal{C}^2(\mathbb{R}^2,[0,\infty))$
$\qquad \overset{\shortparallel}{|du_i|^2}$

$$\left.\begin{array}{l}\Delta e \ge -A_0 - a e^2 \\ \int_{B_r(0)} e < \underset{\shortparallel}{\hbar} \\ \qquad\quad \hbar(a)>0\end{array}\right\} \Rightarrow r^2 e(0) \le C\Big(\int_{B_r(0)} e + A_0 r^4\Big)$$

$\Rightarrow$ If $\underset{p_i\to p}{|du_i(p_i)|^2} = e_i(0) \to\infty$ pick $r_i\to 0$ s.t. $r_i^2 e_i(0) > C E(u_i)$

then conclusion wrong, hence energy $\int_{B_{r_i}(p)} |du_i|^2 \ge \hbar$ concentrates at p.

Exercise: Use only Removable Singularity and mean value inequality to prove compactness of lowest energy moduli space.