5 – Transversality

Note Title 8/7/2009

Fix (Σ, j), (M,ω), $A \in H_2(M)$.

Goal: The set of regular $J \in \mathcal{J}(M,\omega)$ resp. $(J_t) \in \mathcal{C}^\infty([0,1], \mathcal{J}(M,\omega))$

(i.e. those for which at all solutions the linearized operator is surjective)

contains a Baire set (a countable intersection of open dense sets).

(I.e. "generic J are regular".)

This follows by intersection arguments from

Thm: The set of regular $J \in \mathcal{J}^\ell =: \overline{\mathcal{J}(M,\omega)}^{\mathcal{C}^\ell}$ is a Baire set for $\ell \geq \operatorname{index}(\Sigma, A) + 2$.

Plan of Proof:

1) The universal moduli space $\mathcal{M}^{univ} = \{(u,J) \mid J \in \mathcal{J}^\ell, u \in \tilde{\mathcal{M}}(J,A)\}$

is a separable $\mathcal{C}^{\ell-1}$-Banach submanifold of $\underset{W^{1,p}(\Sigma,M)}{\underset{\cap}{\mathcal{B}}} \times \mathcal{J}^\ell$.

2) The projection $\pi : \mathcal{M}^{univ} \to \mathcal{J}^\ell$ is a $\mathcal{C}^{\ell-1}$-map and

for all $(u,J) \in \mathcal{M}^{univ}$ $d_{(u,J)}\pi$ is Fredholm with

$\ker d_{(u,J)}\pi \cong \ker \underset{\text{linearization } D_u \text{ of } \bar\partial_J}{\underset{\shortparallel}{D_u \bar\partial_J}}$, $\quad T_J\mathcal{J}^\ell / \operatorname{im} d_{(u,J)}\pi \cong \mathcal{E}_u / D_u \bar\partial_J$

$D_u\bar\partial_J$ surjective $\Leftrightarrow$ $d_{(u,J)}\pi$ surjective

J regular $\Leftrightarrow$ J regular value of π

Sard-Smale: $\mathcal{M}, \mathcal{J}$ separable $\mathcal{C}^k$-Banach manifolds, $k \geq 1$

$\pi : \mathcal{M} \to \mathcal{J}$ $\mathcal{C}^k$ Fredholm map, index $\pi \leq k-1$
(each $d_m\pi$ Fredholm)

$\Rightarrow$ The set of regular values $\{J \in \mathcal{J} \mid \pi(m) = J \Rightarrow d_m\pi \text{ surjective}\}$

is a Baire set in $\mathcal{J}$.

Proof of 2) The projection $\pi : \mathcal{M}^{univ} \to \mathcal{J}^\ell$ is a $\mathcal{C}^{\ell-1}$-map and

for all $(u,J) \in \mathcal{M}^{univ}$ $\quad d_{(u,J)}\pi$ is Fredholm with

$\ker d_{(u,J)}\pi \cong \ker D_u\bar{\partial}_J \qquad T_J\mathcal{J}^\ell / \operatorname{im} d_{(u,J)}\pi \cong \mathcal{E}_u / D_u\bar{\partial}_J$

uses

Lemma: $D : B \to E$ Fredholm, $L : J \to E$ bounded linear

If $D \oplus L : B \times J \to E$ is onto then $\pi : \ker(D \oplus L) \to J$ is

Fredholm with $\ker \pi \cong \ker D$, $J/\operatorname{im}\pi \cong E/\operatorname{im} D$ $(\Rightarrow \operatorname{ind}\pi = \operatorname{ind} D)$.

Indeed:

$T_{(u,J)}\mathcal{M}^{univ} = \ker\left(D_u\bar{\partial}_J + L_{(u,J)}\right)$ for some $L_{(u,J)} : T_J\mathcal{J}^\ell \to \mathcal{E}_u$

$D_u\bar{\partial}_J + L_{(u,J)} : T_u\mathcal{B} \times T_J\mathcal{J}^\ell \to \mathcal{E}_u$ surjective by 1.)

$d_{(u,J)}\pi : T_{(u,J)}\mathcal{M}^{univ} \to T_J\mathcal{J}^\ell$ is the projection π in Lemma.

Proof of 1.)

$$\begin{array}{c}\mathcal{E}\\ \downarrow \;\big\uparrow s(u,J)=\bar\partial_J u\\ \mathcal{B}\times\mathcal{J}^\ell\end{array}$$ is a $C^{\ell-1}$-section transverse to 0.

- Why (only) $C^{\ell-1}$? Consider $f: W^{1,p}\to L^p$, $u\mapsto J(u)\circ du$

  Its k-th derivative contains $\xi\mapsto \underbrace{\nabla^k J(u)}_{C^0 \text{ if } k\le\ell}(\xi,\dots\xi)\circ du$

  Transition maps in $\mathcal{E}$ are only $C^{k\le\ell-1}$ since parallel transport depends on ∇J.

- Why is $D_{(u,J)}s : T_u\mathcal{B}\times T_J\mathcal{J}^\ell \longrightarrow \mathcal{E}_u$ onto?

$$(\xi, Y) \longmapsto (D_u\bar\partial_J)\xi + \underbrace{\tfrac12 Y(u)\circ du\circ j}_{=L_{(u,J)}Y}$$

$\to$ it has closed image since $\underset{\text{closed, finite codim.}}{\mathrm{im}\, D_u\bar\partial_J} \subset \mathrm{im}\, D_{(u,J)}s \subset \mathcal{E}_u$

$\to$ to check image is dense suppose $\eta\in(\mathrm{im}\, D_{(u,J)}s)^\perp\subset\mathcal{E}_u^*$

That is $\eta\in L^q(\Sigma,\Lambda^{0,1}T\Sigma\otimes u^*TM)$; $\frac1q+\frac1p=1$ satisfies

(a) $\int_\Sigma\langle\eta, D_u\xi\rangle = 0 \qquad \forall \xi\in W^{1,p}(\Sigma,u^*TM)$

$$\Rightarrow D_u^*\eta=0 \xRightarrow{\text{elliptic regularity}} \eta\in W^{1,p}\hookrightarrow C^0$$

(b) $\int_\Sigma\langle\eta, Y(u)\circ du\circ j\rangle = 0 \qquad \forall Y\in C^\ell(M, \mathrm{End}(TM,J,\omega))$

$\Rightarrow \eta(z_0)=0 \quad \forall z_0$ injective point of u $\qquad \begin{pmatrix} YJ+JY=0\\ \omega(Y\cdot,\cdot)=-\omega(\cdot,Y\cdot)\end{pmatrix}$

$\Rightarrow \eta\equiv0$